Subquadratic diameter computation in graphs of bounded VC-dimension

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Aalto University
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Super-quadratic distance problems

Given a simple undirected graph G on n vertices and m edges, and the corrsponding shortest-path metric,

- **Diameter**
  Find the maximum distance among vertices
- **Eccentricites**
  For each vertex v find the minimum radius of ball at v that covers G.
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Lower bound by Roditty and V. Williams (2013):
No $O(n^{2-\varepsilon})$ algo to decide diameter 2 vs 3 in sparse graphs under SETH.

SotA sub-quadratic diameter computation

First breakthrough:
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Graphs of distance VC-dimension d: $\widetilde{O}(Dmn^{1-1/d})$ [DHV22, DKP24]

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Unit disk graphs with $+\mathbf{2}$ **additive error**: $\widetilde{O}(n^{2-1/18})$ [CGL24]

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$\Omega(n^{2-\varepsilon})$ lower bound for many intersection graphs [B**K**KNP22]:
Unit balls, axis-aligned unit cubes, unit segments, etc.

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Is there a subquadratic algorithm
for diameter in unit disk graphs?

Results: Subquadratic algorithms for diameter

graph class	best previous		new
planar	$\widetilde{O}(n^{5/3})$	[Cab18, GKM+21]	
K_h-minor-free	$\widetilde{O}(n^{2-1/(3h-1)})$	[DHV22, LW24]	$\widetilde{O}(n^{2-1/(2h-2)})$
VC-dim.-bounded	$\widetilde{O}(\min\{Dmn^{1-1/d},\ mn\})$	[DHV22, DKP24]	$\widetilde{O}(mn^{1-1/(2d)})$
unit square	$\widetilde{O}(\min\{Dn^{7/4},\ n^2\})$	[DKP24]	$O^*(n^{2-1/8})$
arbitrary square	$\widetilde{O}(n^2)$	[CS19]	$\widetilde{O}(n^{2-1/12})$
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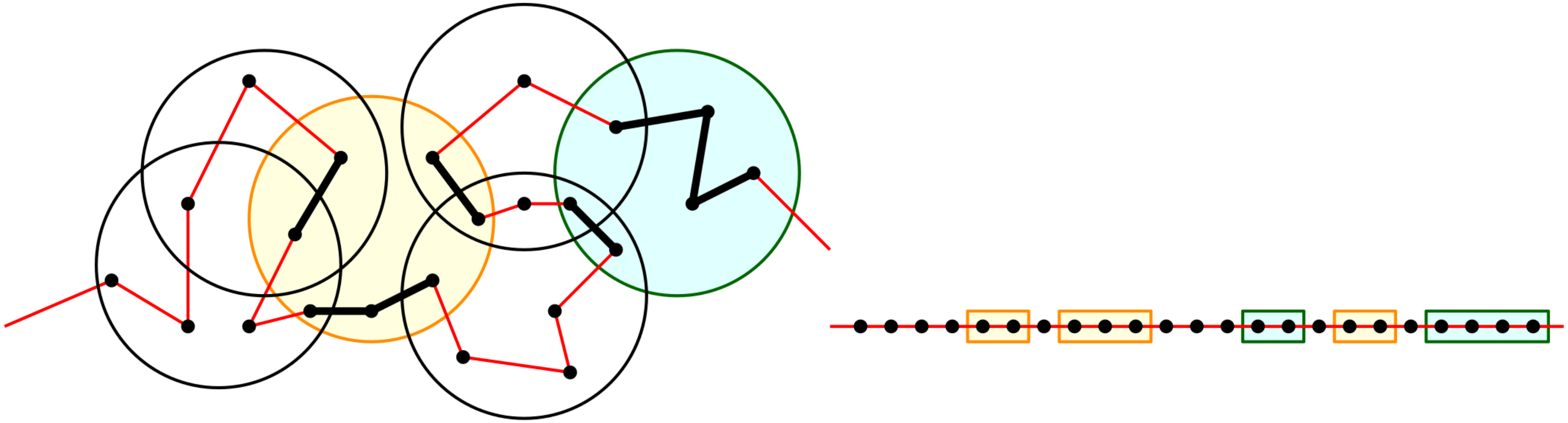
- We don't need separators!
- More efficient ball growing process based on very simple LDD
- New data structures and ideas for geom. intersection graphs

Implicit representations



Good implicit representation via stabbing paths?

Diameter $\leq D$ iff all balls in $\mathcal{N}^D := \{N^D(v) \mid v \in G\}$ cover $V(G)$.
Strategy: grow balls to compute all balls $\mathcal{N} = \bigcup_r \mathcal{N}^r$.

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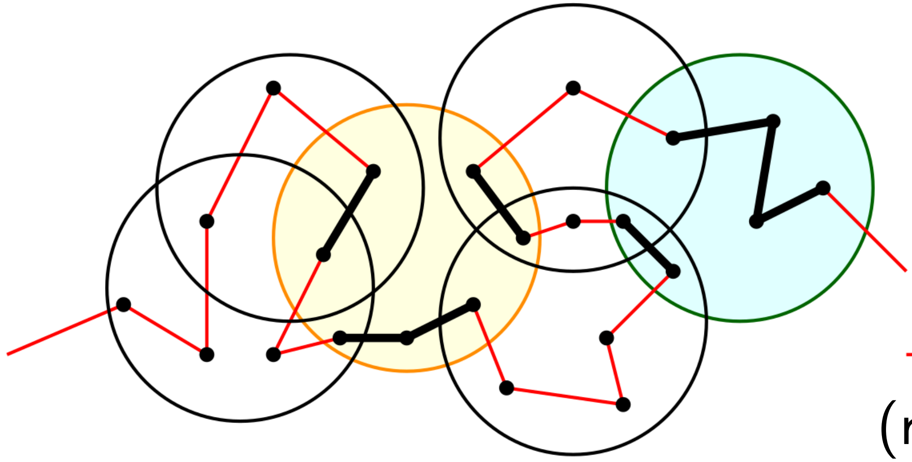

(recall segment tree)

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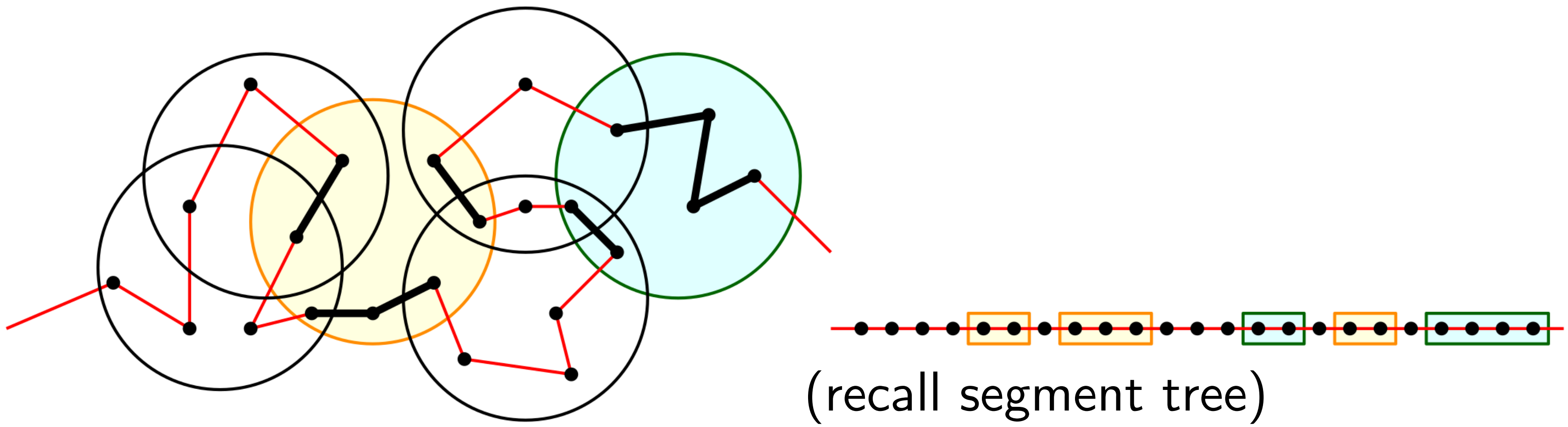

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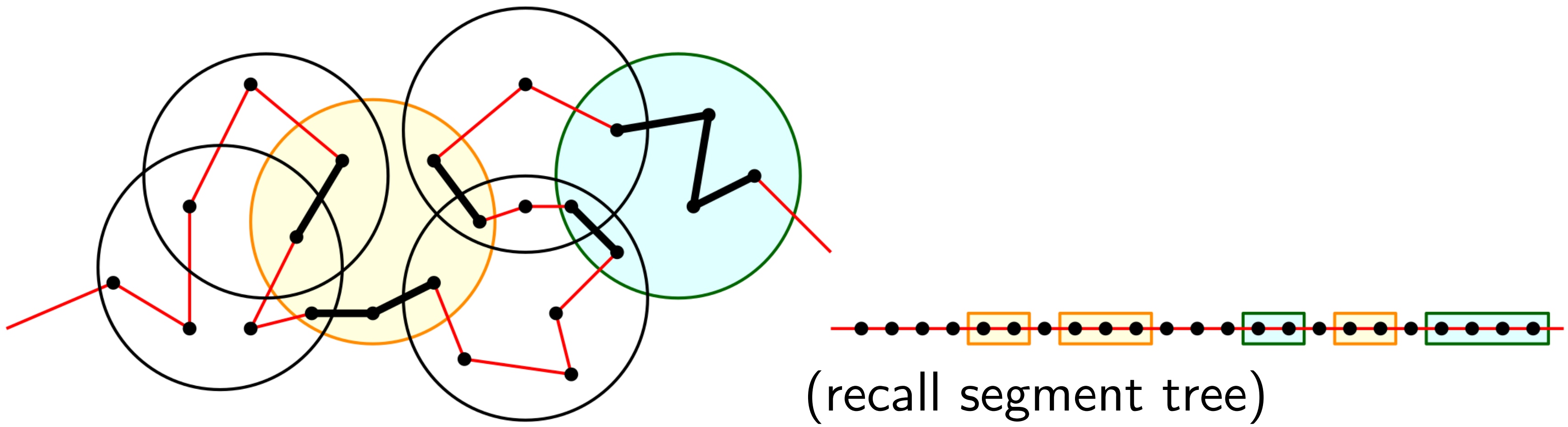

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We want subquadratic representation in amortized sense:

$$\sum_{X \in \mathcal{N}} |\text{Rep}_\lambda(X)| = O(n^{1.999}).$$

Distance VC dimension in graphs

Given a set family $\mathcal{X}$ over universe V, the set $A \subset V$ is *shattered* by $\mathcal{X}$ if:
for all $S \subset A$ there is $X \in \mathcal{X}$ s.t. $X \cap A = S$.

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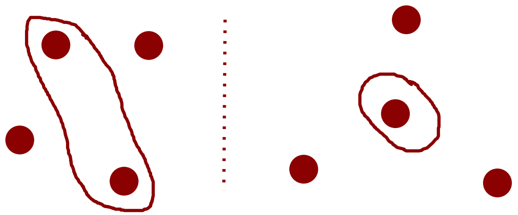

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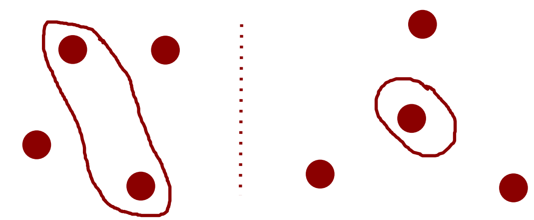

$G :=$ undirected unweighted graph, $N^r(v) =$ ball of radius r at $v \in V(G)$.

The *distance VC-dimension* d *of* G is the VC dimension of $\mathcal{N} := \bigcup_r \mathcal{N}^r(v)$.

Stabbing paths and VC-dimension

Lemma (Informal)
Given* a set system $(V, \mathcal{N})$ of VC-dimension** d we can construct a stabbing path λ in $\widetilde{O}(n^{1+1/d})$ time such that with high*** probability:

$$\sum_{X \in \mathcal{N}} |\mathsf{Rep}_\lambda(S)| = \widetilde{O}(|\mathcal{N}| n^{1-1/d})$$

Constructed via recursive sampling.

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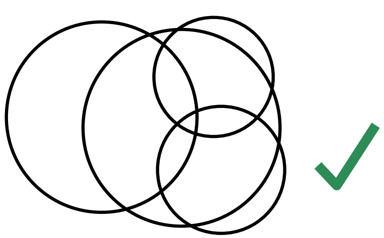
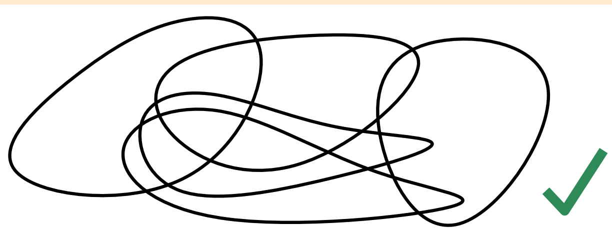
Theorem (adapted from [CGL24])

Intersection graphs of pseudo-disks have distance VC dimension at most 4.





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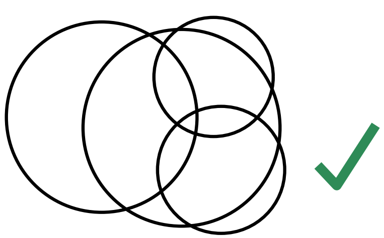

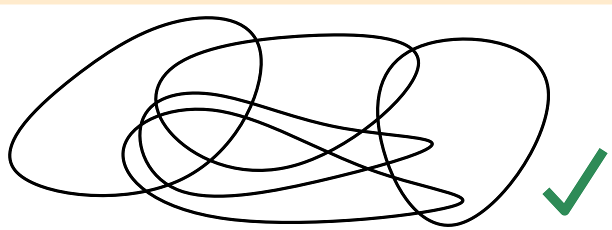

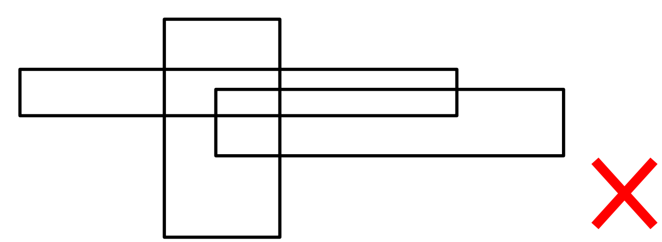

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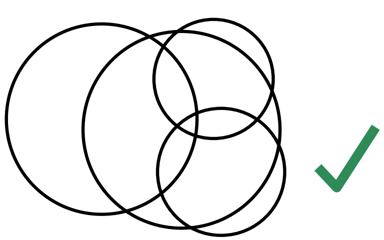
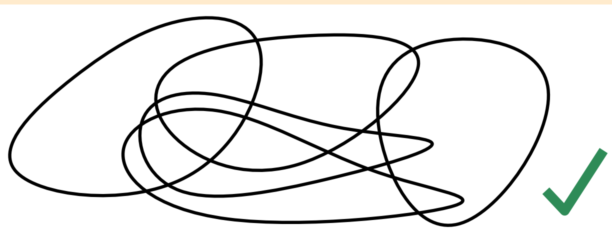
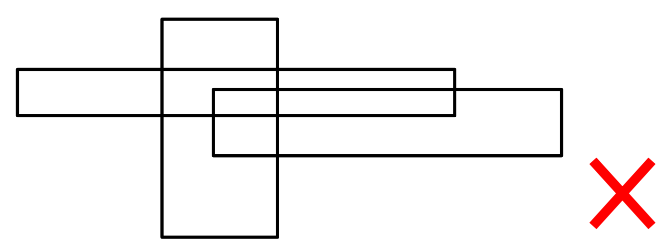
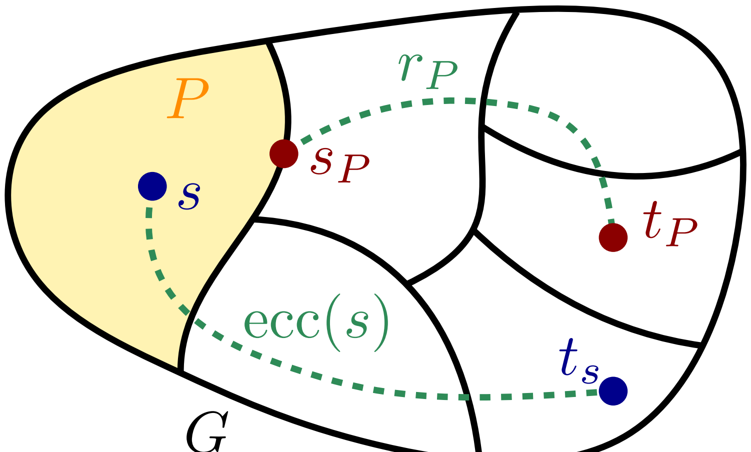






Generic framework and data structures



Generic framework: LDD + ball growing

Step 1. Fix Δ, decompose G into $\widetilde{O}(n/\Delta)$ pieces P of (strong) diameter $\leq \Delta$ s.t. total boundary is small: $\sum_P |\partial P| = \widetilde{O}(n/\Delta)$.

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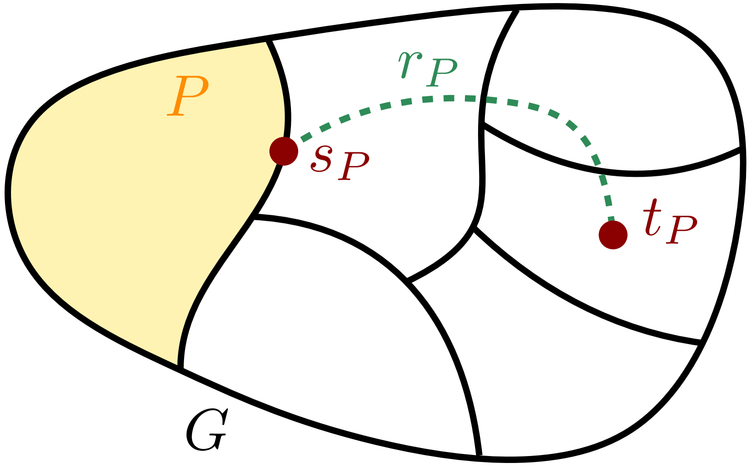

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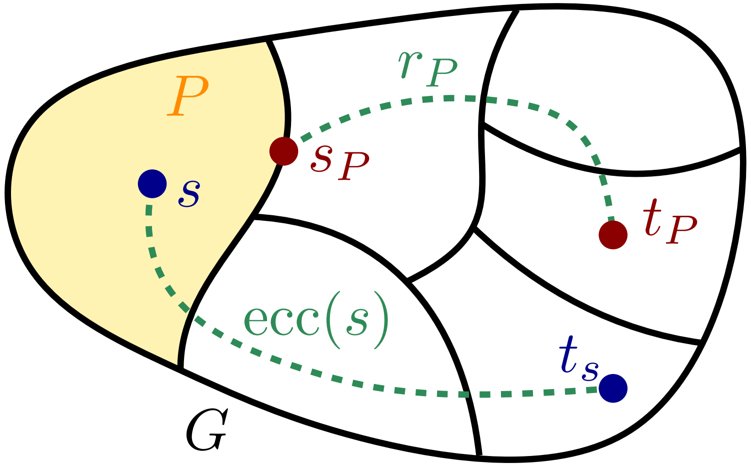
By triangle-ineq:
$\mathrm{dist}_G(s_P, t) - \Delta \leq \mathrm{dist}_G(s, t) \leq \mathrm{dist}_G(s_P, t) + \Delta$



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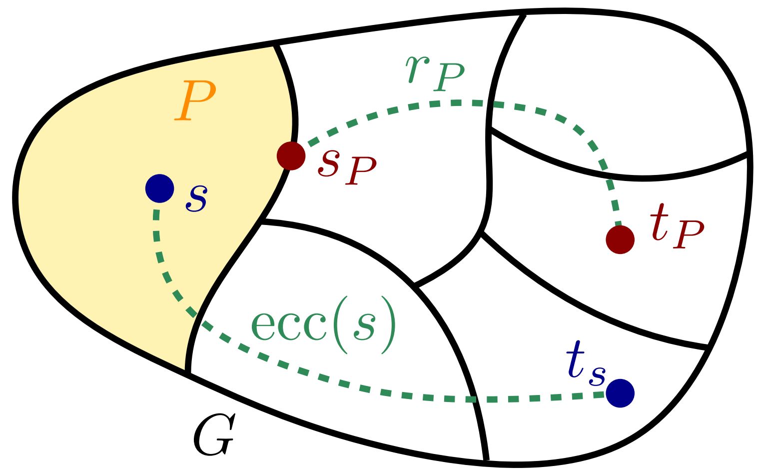

The *relevant region of P* is

$$R_P := \{v \in G \mid \mathrm{dist}_G(s_P, v) \geq r_P - 2\Delta\}.$$

The *modified r-ball* of $s \in P$ is $\widehat{N}^r(s) := N^r(s) \cap R_P$. Let $\widehat{\mathcal{N}} := \{\widehat{N}^r(s)\}_{r,s}$.

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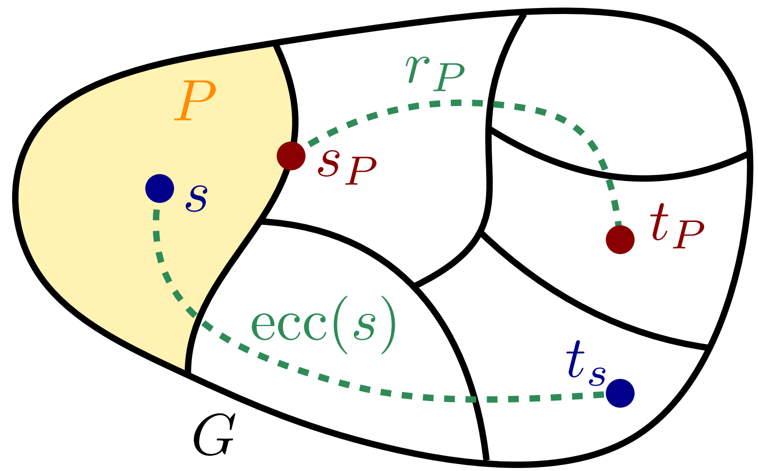

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Lemma

The modified ball system $\widehat{\mathcal{N}}$ has the same VC-dim as $\mathcal{N}$ and $|\widehat{\mathcal{N}}| = O(\Delta n)$

Step 3. Compute stabbing path λ for $\widehat{\mathcal{N}}$.
For each P and each $s \in P$, from $r = r_P - 3\Delta - 1$ to $r = r_P + \Delta$, compute $\mathrm{Rep}_\lambda(\widehat{N}^r(s))$.

Efficient ball growing?

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$$\mathsf{Rep}_\lambda(\widehat{N}^{r+1}(s)) = \bigcup_{v \in N(s)} \mathsf{Rep}_\lambda(\widehat{N}^r(v))$$

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We need*: $\widetilde{O}$(input size) preprocessing and $\widetilde{O}$(output size) query time

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Hard and non-decomposable range searching...

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Given objects $\mathcal{O}$ and $C : \mathcal{O} \to [n]$ (color), design a data structure that:
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Theorem

If we can construct in $\widetilde{O}(|\mathcal{O}_{RC}|)$ time a data structure $\mathcal{D}_{RC}$ with $\widetilde{O}(1)$ query for DSP3, then for any $b \in [1, n]$, we can construct a data structure for DSP2 with total run time $\widetilde{O}(N_{IC} \cdot b + L_{IC}/b)$.

Data structure for squares

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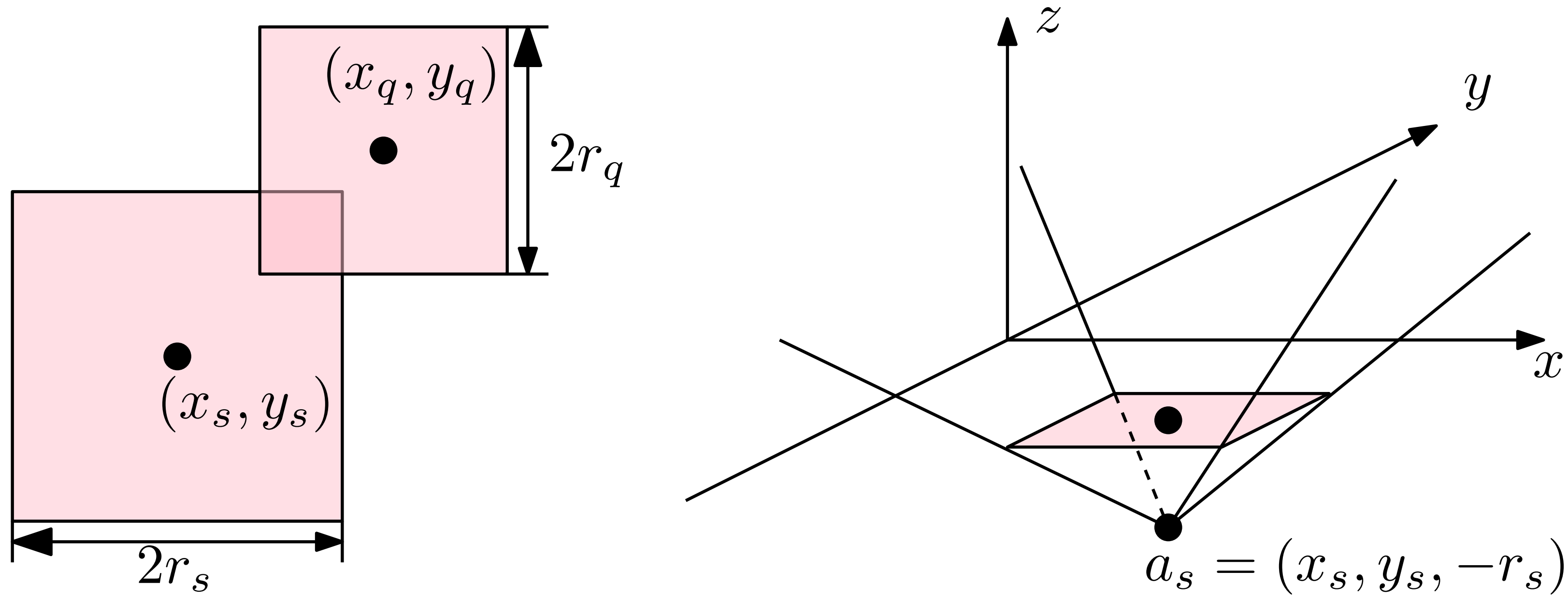

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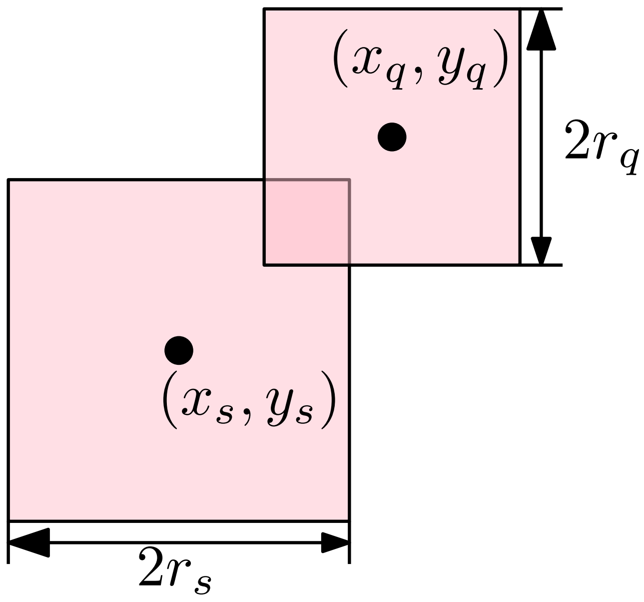

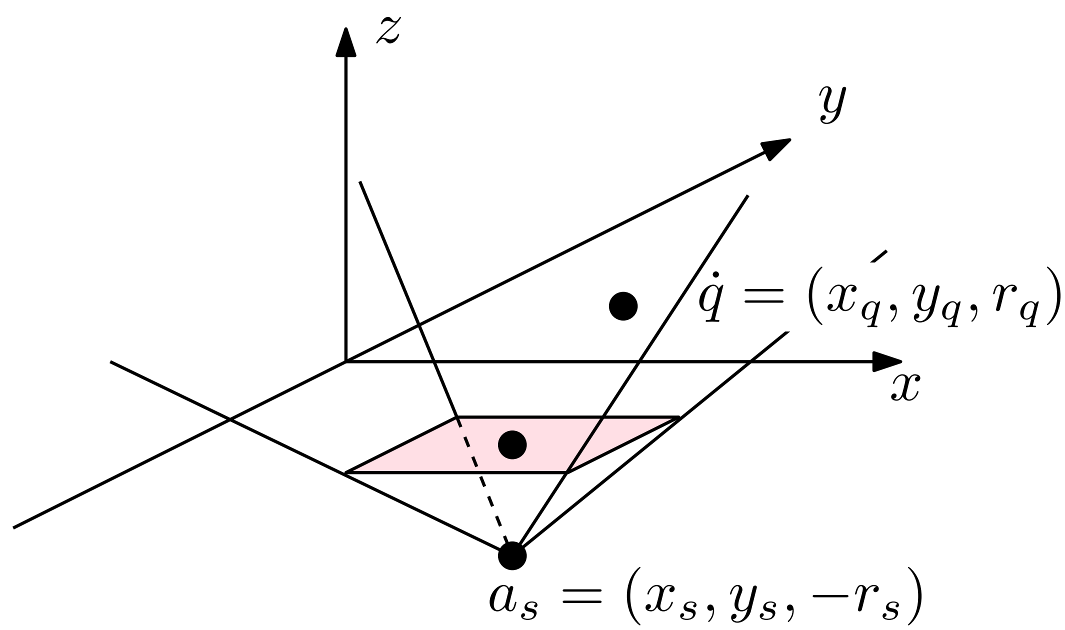

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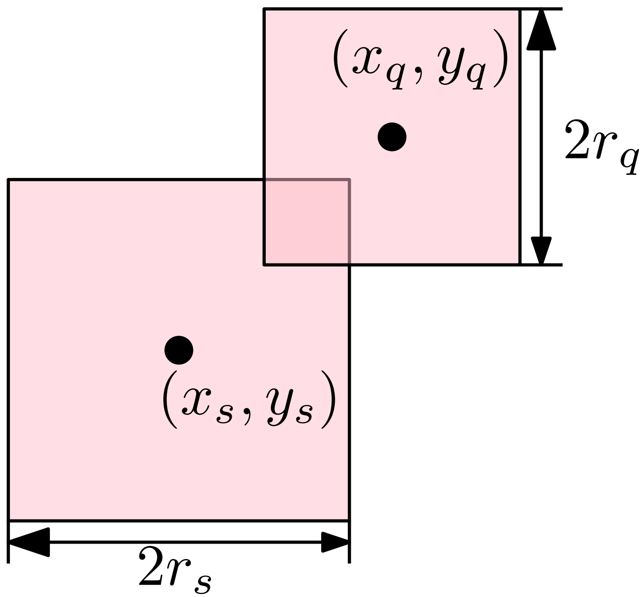

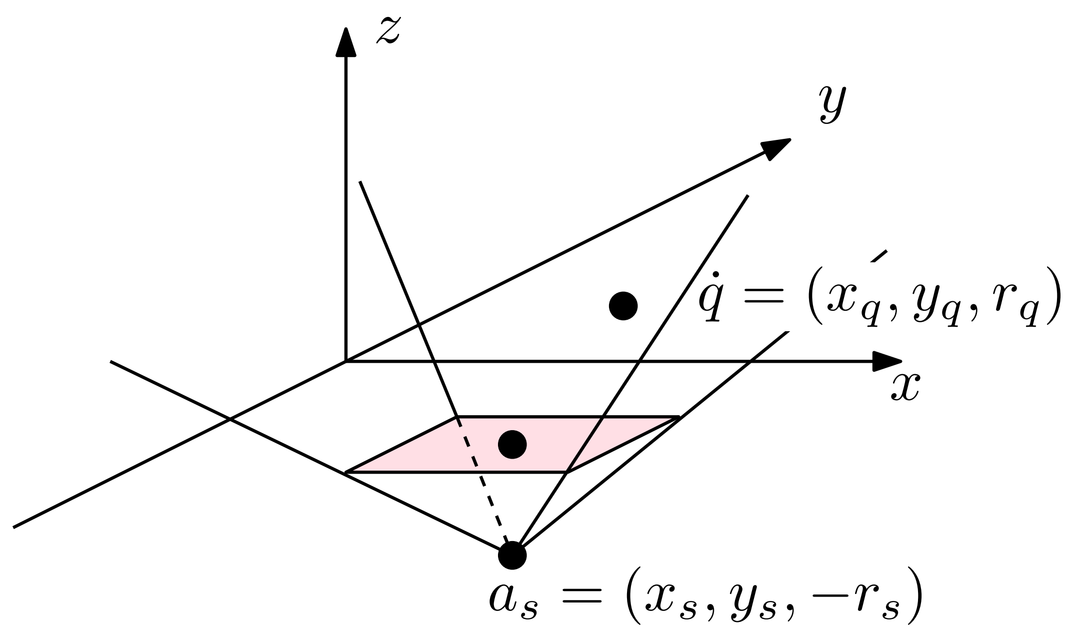

Rough idea:
- q has intersection with color class i iff $\dot{q}$ is above lower envelope of color-i cones.

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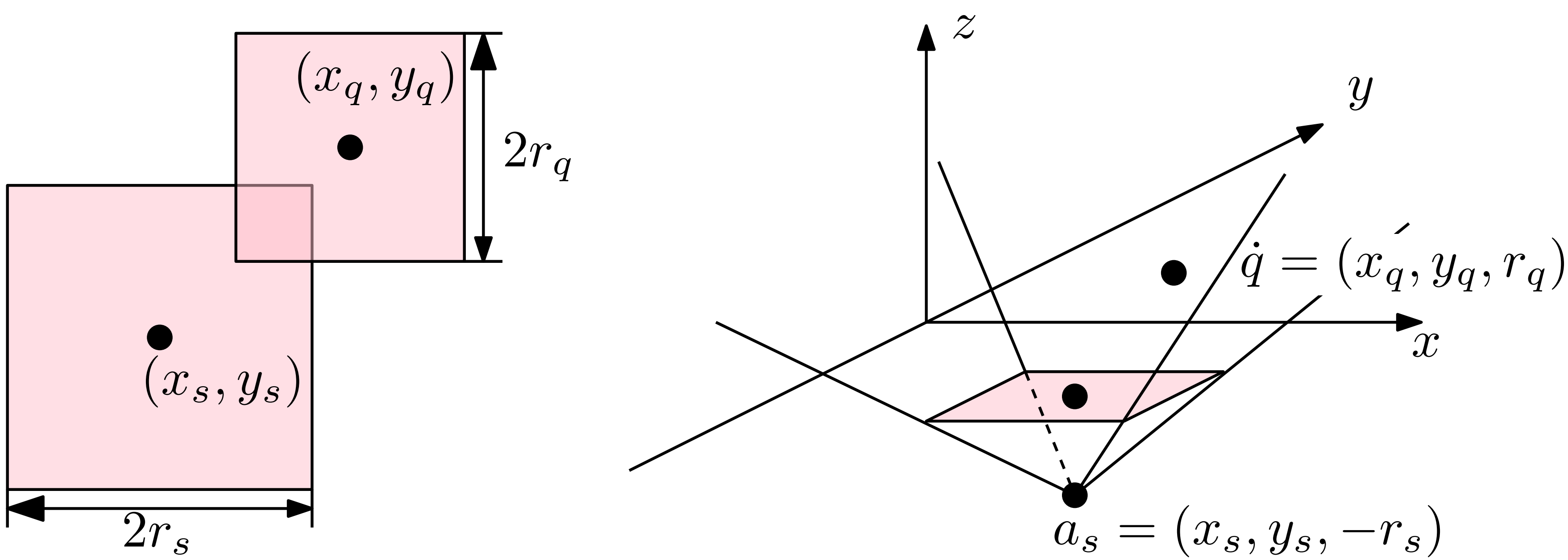

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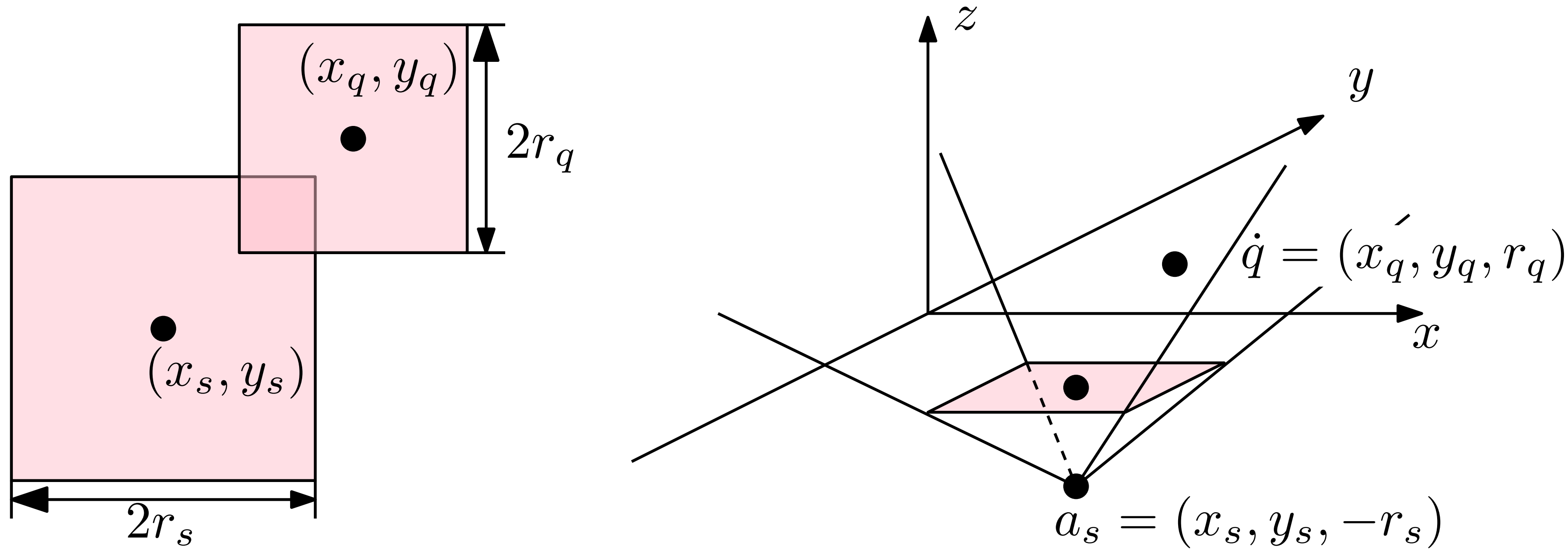

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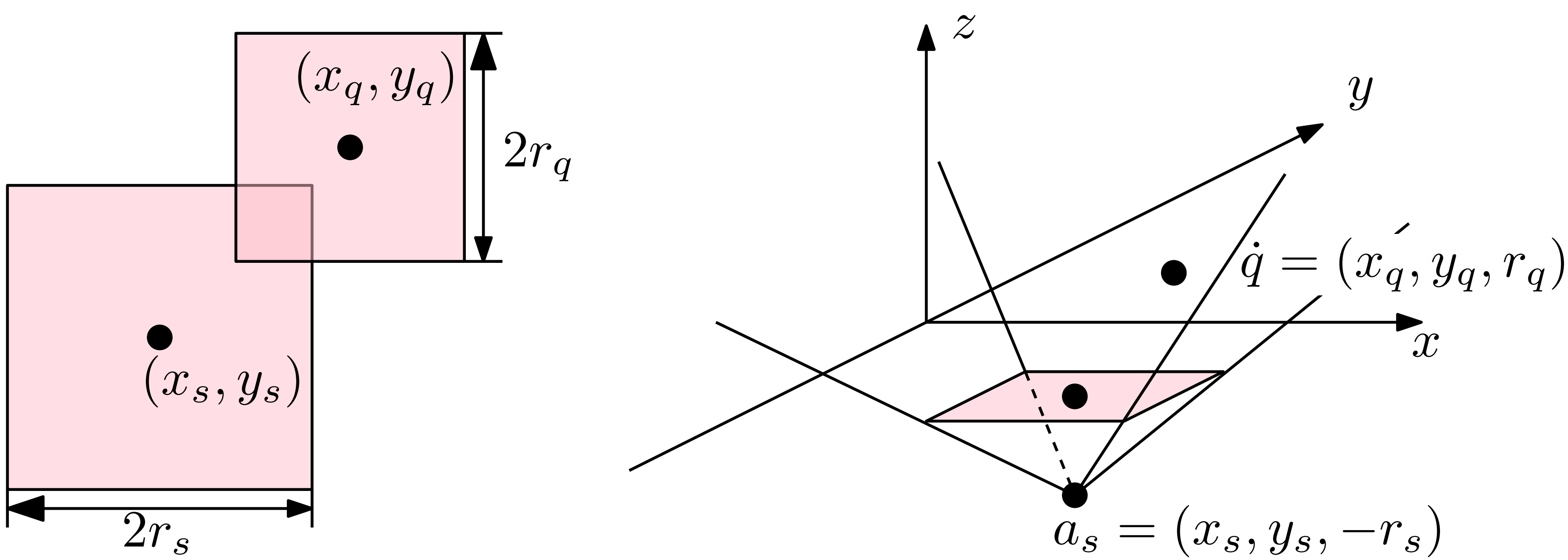

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- build (reverse) range counting data structure on all slabs

Running time analysis for squares

Stabbing path λ gives:

$$\sum_{P} \sum_{s \in P} \sum_{r=r_P-2\Delta}^{r_P+\Delta} |\mathsf{Rep}_\lambda(\hat{N}^r[s])| \;=\; \widetilde{O}(\Delta \cdot n^{2-1/d}) = \widetilde{O}(\Delta \cdot n^{7/4}) \quad (d=4)$$

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BFSes on ∂P → n^2/Δ

Constructing λ → $n^{5/4}$

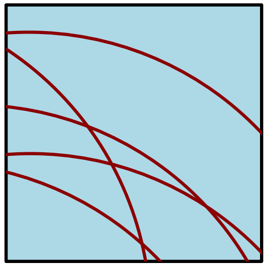
$$\widetilde{O}(n^2/\Delta + n^{5/4}) + \sum_{P}\sum_{r=r_P-2\Delta}^{r_P+\Delta}\widetilde{O}\left(b\cdot\sum_{s\in P}\Big((|\mathsf{Rep}_{\lambda_P}(\hat{N}^{r-1}[s])| + |\mathsf{Rep}_{\lambda_P}(\hat{N}^{r}[s])|\Big) + |P|n\right.$$

$$= \widetilde{O}(n^2/\Delta + n^{5/4}) + \widetilde{O}(b\Delta\cdot n^{7/4}) + \widetilde{O}(n^2\Delta/b) \quad (d=4)$$

$$= \widetilde{O}(n^{2-1/16}). \quad \text{(for optimal choices of } b=\Delta^2 \text{ and } \Delta = n^{1/16})$$

Unit disks: types, resamples

Base data strucutre problem is related to Hopcroft's problem, $\Omega(n^{1/3})$ query time likely.



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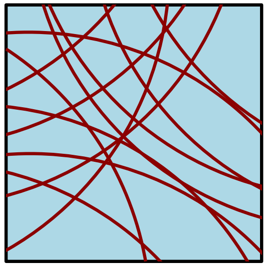

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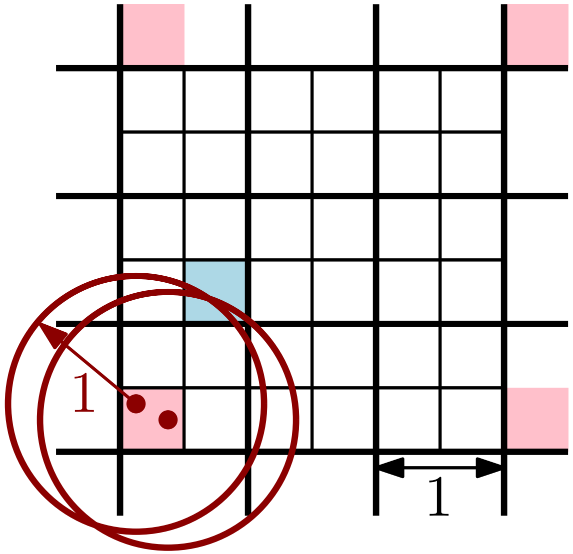

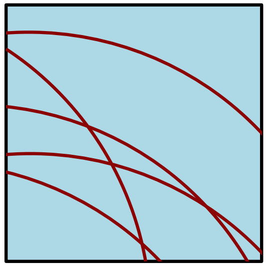

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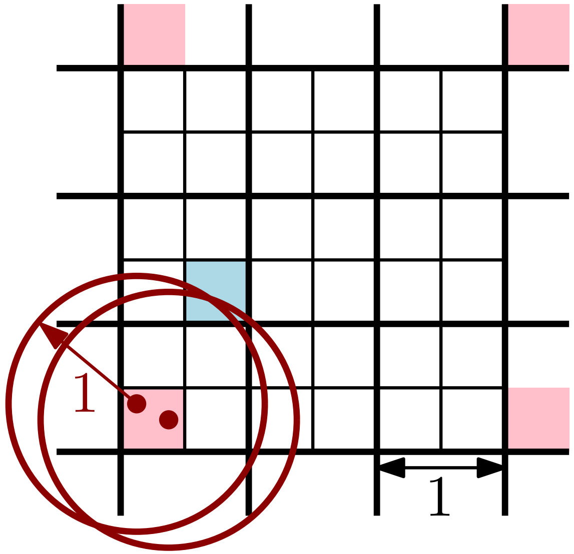

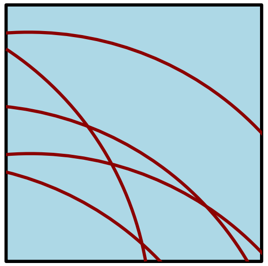

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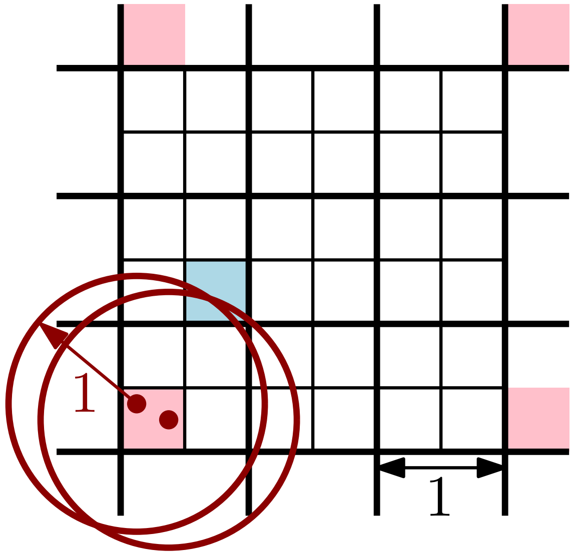

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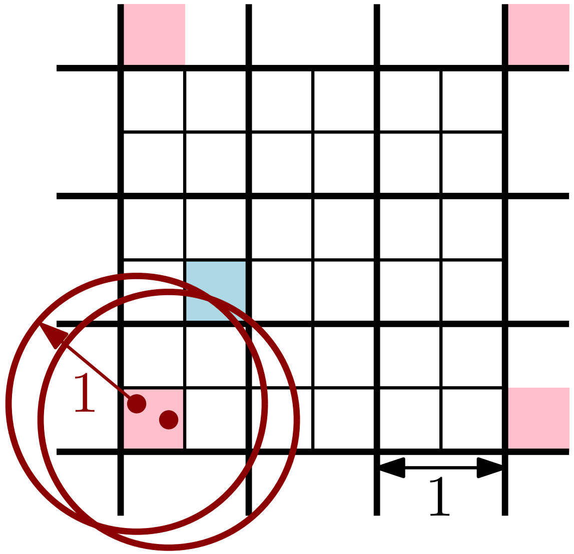

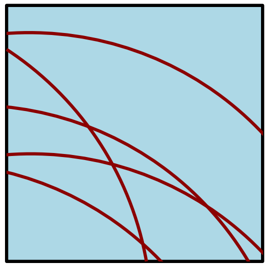

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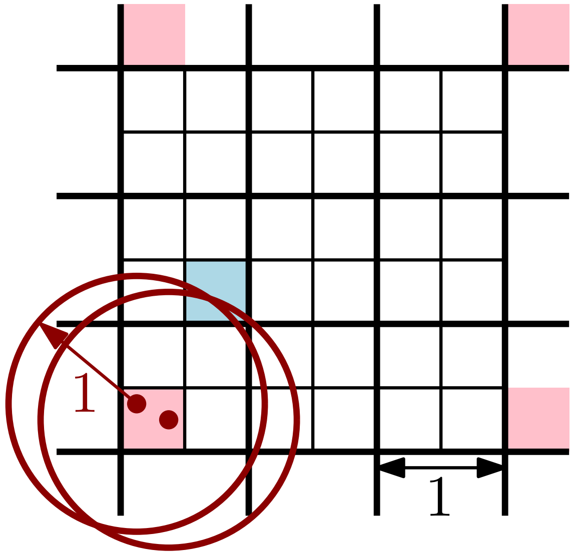

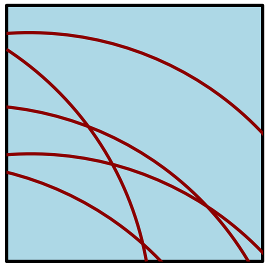

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- Switching stabbing paths is too costly if pieces are small.
  $\rightarrow$ Work only with pieces larger than a threshold; for small pieces, we switch to a different algo (based on distance compression)

Ongoing work, a conjecture, and open problems

VC-dim d + efficient interval cover DS
$\rightsquigarrow O(n^{2-f(d)})$ diameter computation/distance oracle

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There is some $f(d) > 0$ such that if $\mathcal{G}$ is a (nice?) graph class with distance VC-dim$\leq d$, then there is a $O(n^{2-f(d)})$ time algorithm for diameter.

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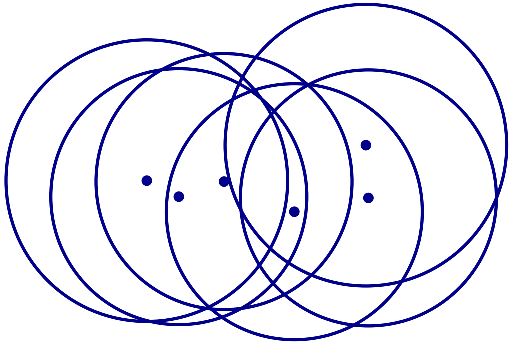

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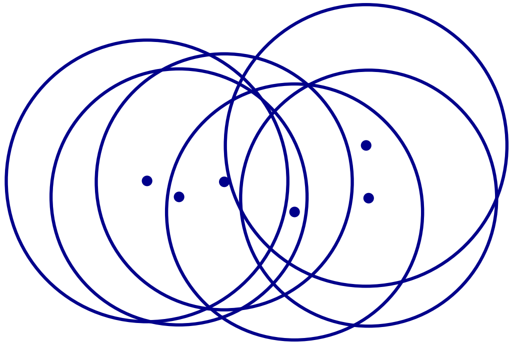

Thanks for listening!

Diameter:

graph class	best previous		new
planar	$\widetilde{O}(n^{5/3})$	[Cab18, GKM+21]	
K_h-minor-free	$\widetilde{O}(n^{2-1/(3h-1)})$	[DHV22, LW24]	$\widetilde{O}(n^{2-1/(2h-2)})$
VC-dim.-bounded	$\widetilde{O}(\min\{Dmn^{1-1/d},\ mn\})$	[DHV22, DKP24]	$\widetilde{O}(mn^{1-1/(2d)})$
unit square	$\widetilde{O}(\min\{Dn^{7/4},\ n^2\})$	[DKP24]	$O^*(n^{2-1/8})$
arbitrary square	$\widetilde{O}(n^2)$	[CS19]	$\widetilde{O}(n^{2-1/12})$
unit disk	$O(n^2\sqrt{\frac{\log\log n}{\log n}})$	[CS16]	$O^*(n^{2-1/18})$

Distance oracle
(construction time/space)
Query: $\widetilde{O}(1)$.

graph class	best previous		new
planar	$n^{3/2+o(1)}$, $n^{1+o(1)}$	[CGL+23]	
K_h-minor-free	$\widetilde{O}(n^{2-1/(3h-1)})$	[LW24]	
VC-dim.-bounded	$O(mn)$, $O(n^2)$	folklore	$\widetilde{O}(mn^{1-1/(4d+1)})$
unit square	$\widetilde{O}(n^2)$	[CS19]	$O^*(n^{2-1/16})$
arbitrary square	$\widetilde{O}(n^2)$	[CS19]	$\widetilde{O}(n^{2-1/20})$
unit disk	$O(n^2\sqrt{\frac{\log\log n}{\log n}})$	[CS16]	$O^*(n^{2-1/20})$