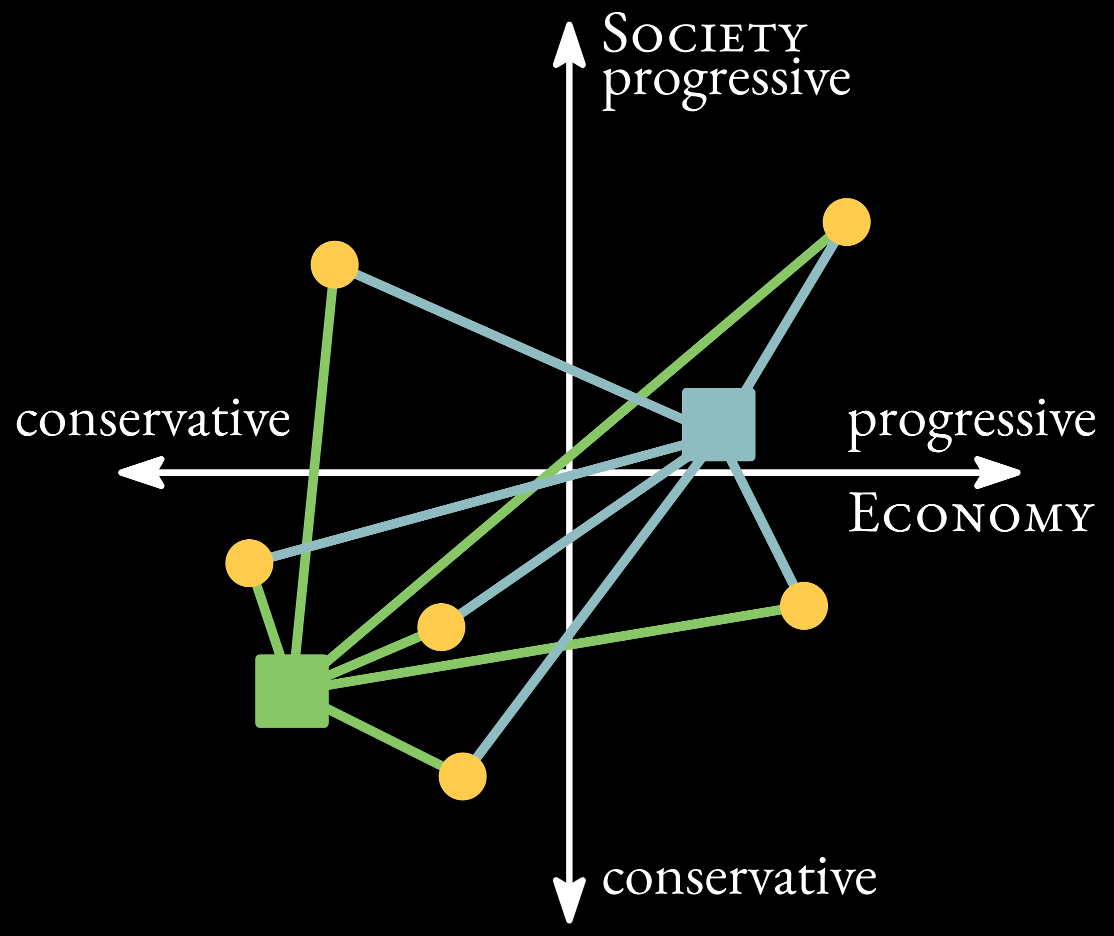

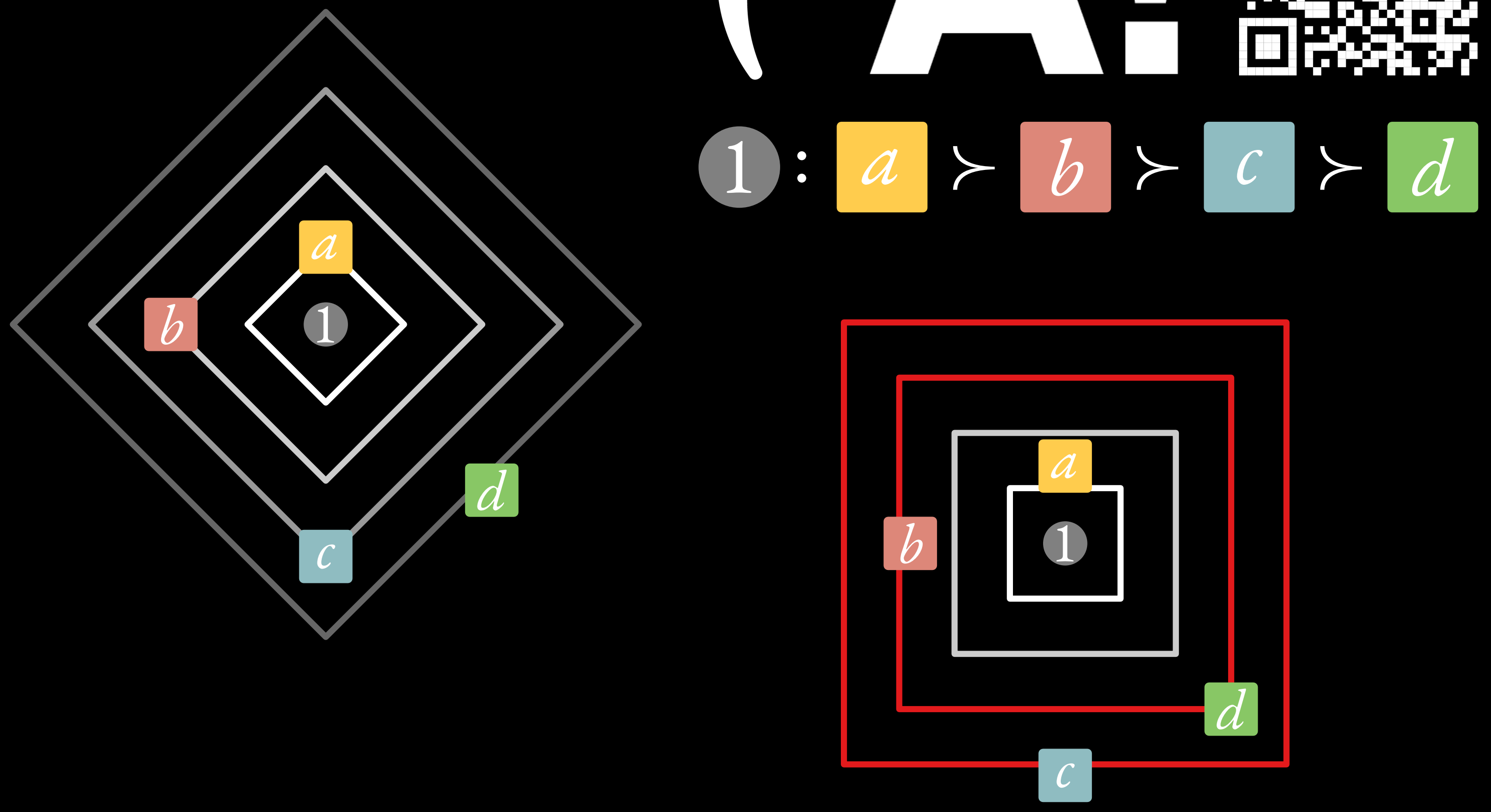

Real Preferences Under Arbitrary Norms

Joshua Zeitlin and *Corinna Coupette*

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Given:

Voters V = 1 2 3

Alternatives A = a b c d

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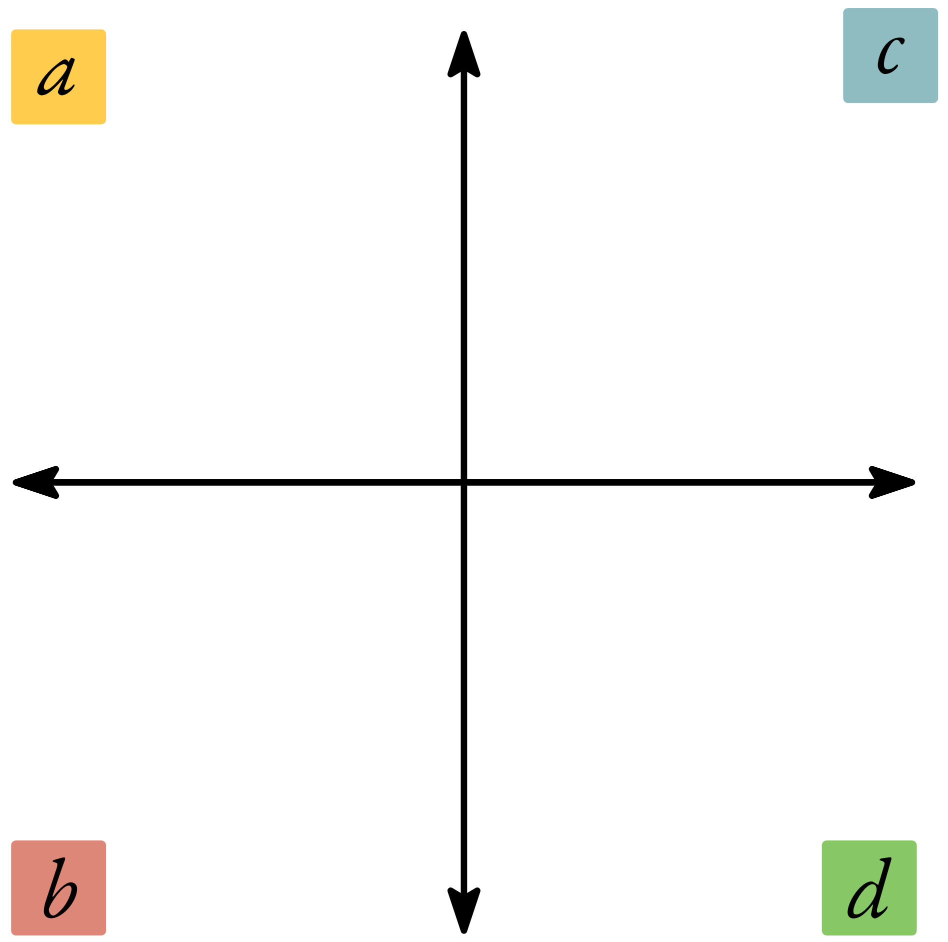

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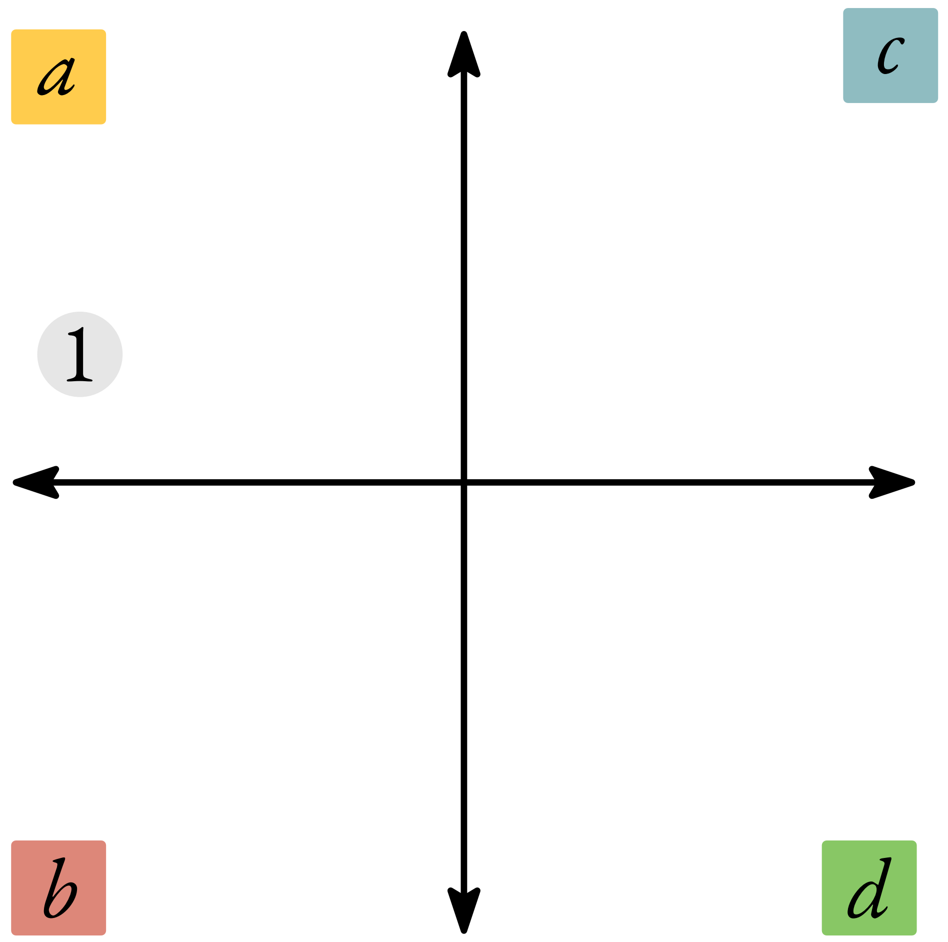

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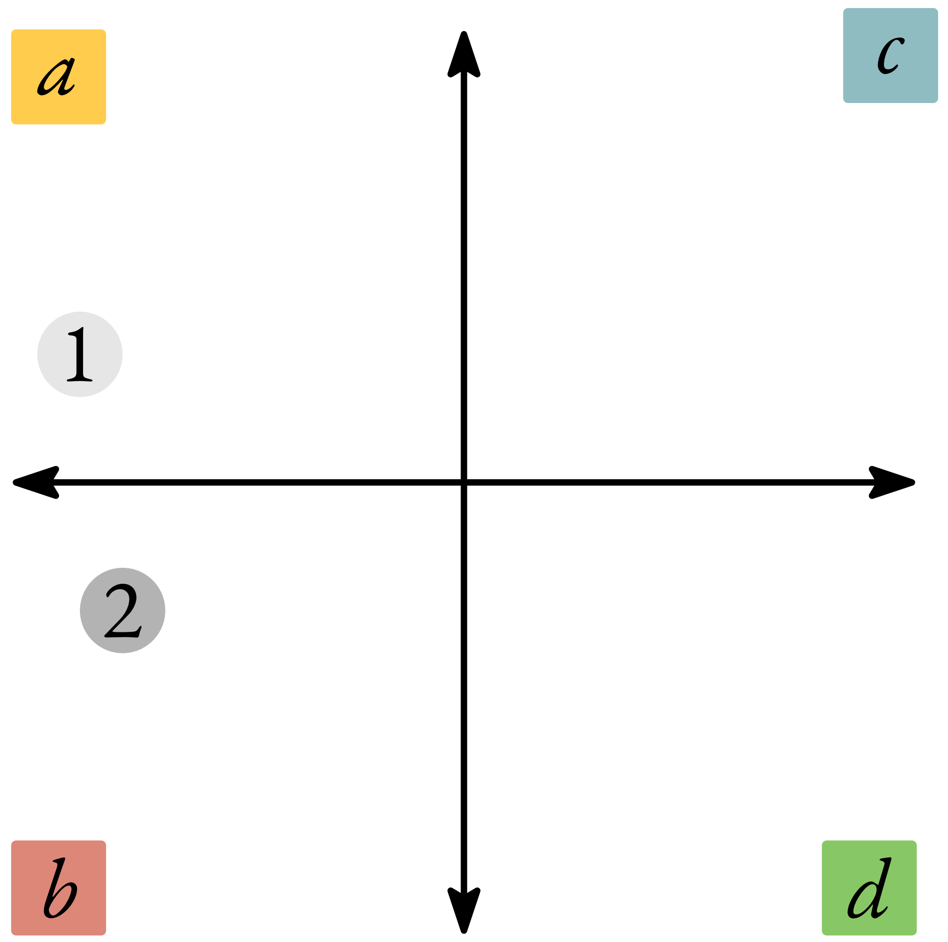

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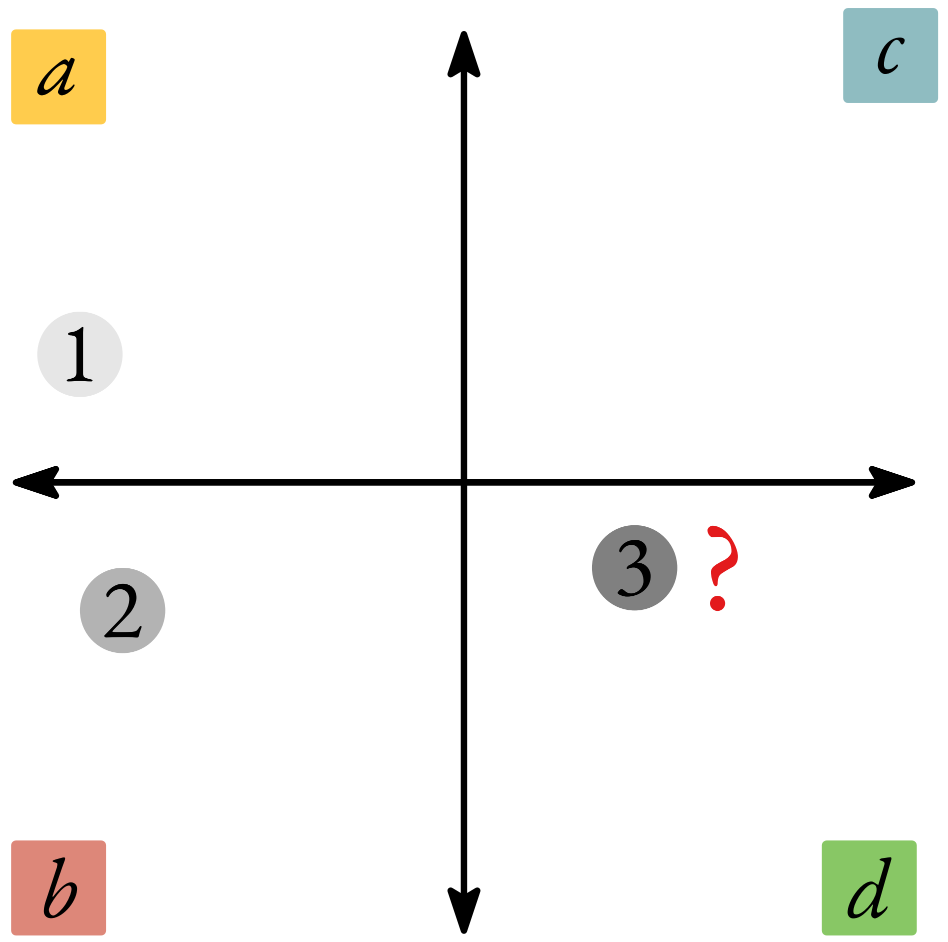

. . . or do they?

Formalization

Given:

Voters $V = \{1, \ldots, n\}$

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Strict preferences $\succ_i$: $a_x \succ a_y \Leftrightarrow i$ prefers a_x over a_y

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Rank-preserving embeddings

Given a preference profile $\mathcal{P}$, a dimension d, and a normed real vector space $(\mathbb{R}^d, \|\cdot\|)$, an assignment of coordinates $\mathbf{a}_j \in \mathbb{R}^d$ to alternatives $a_j \in A$ and coordinates $\mathbf{v}_i \in \mathbb{R}^d$ to voters $i \in V$ constitutes a *rank-preserving embedding* of $\mathcal{P}$ into $(\mathbb{R}^d, \|\cdot\|)$ if $a_j \succsim_i a_k \iff \|\mathbf{v}_i - \mathbf{a}_j\| \leq \|\mathbf{v}_i - \mathbf{a}_k\|$.

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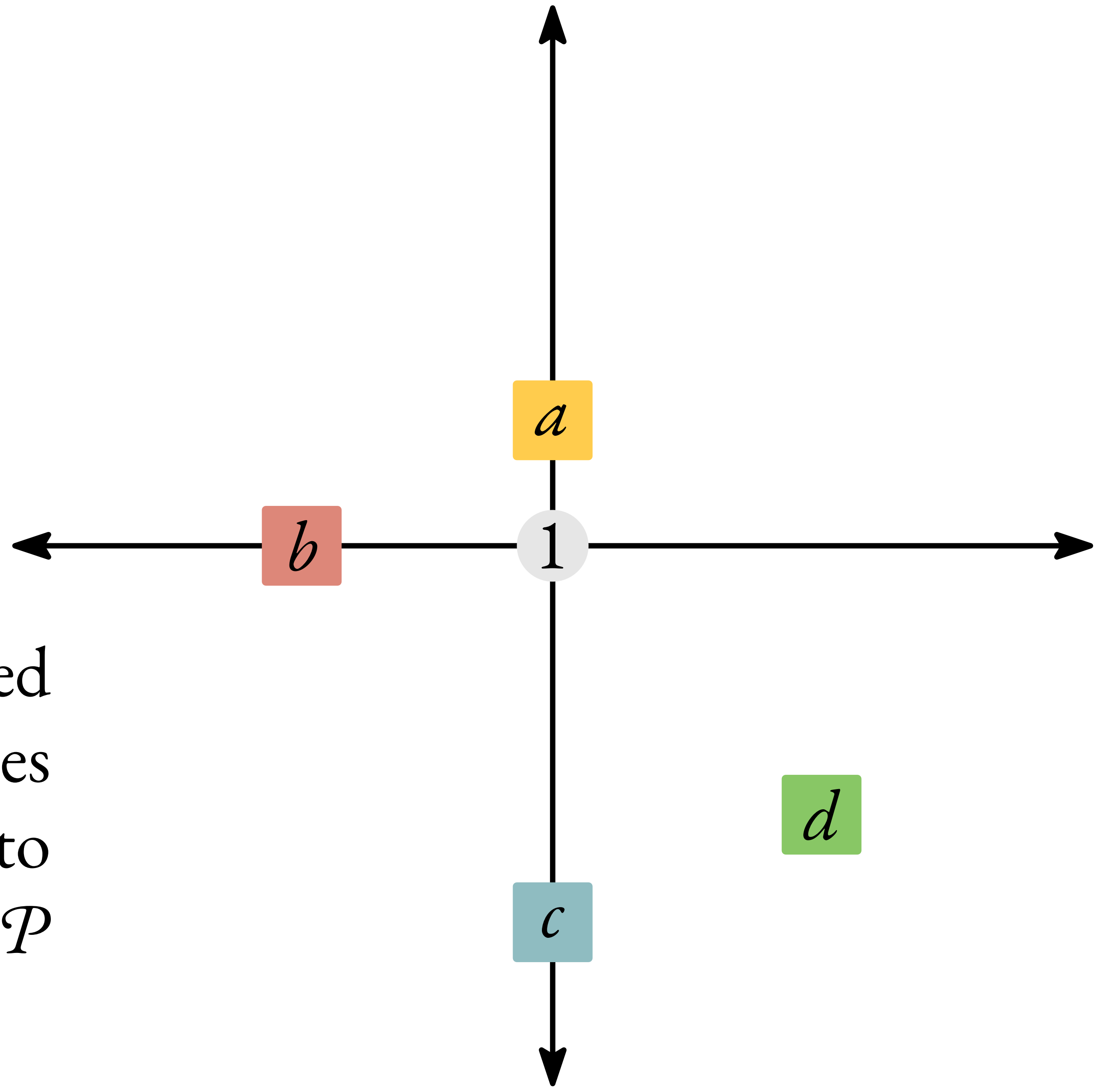

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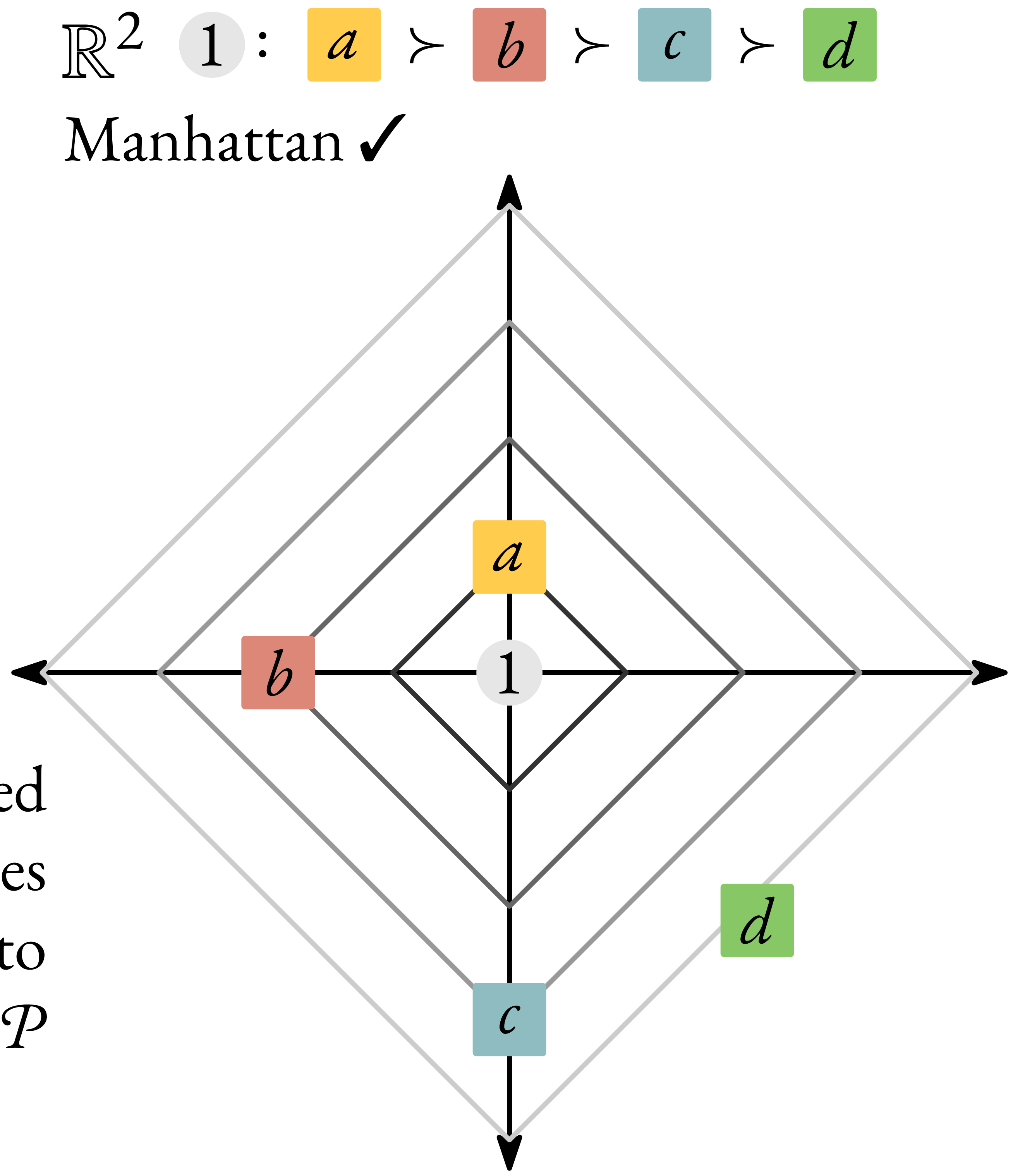

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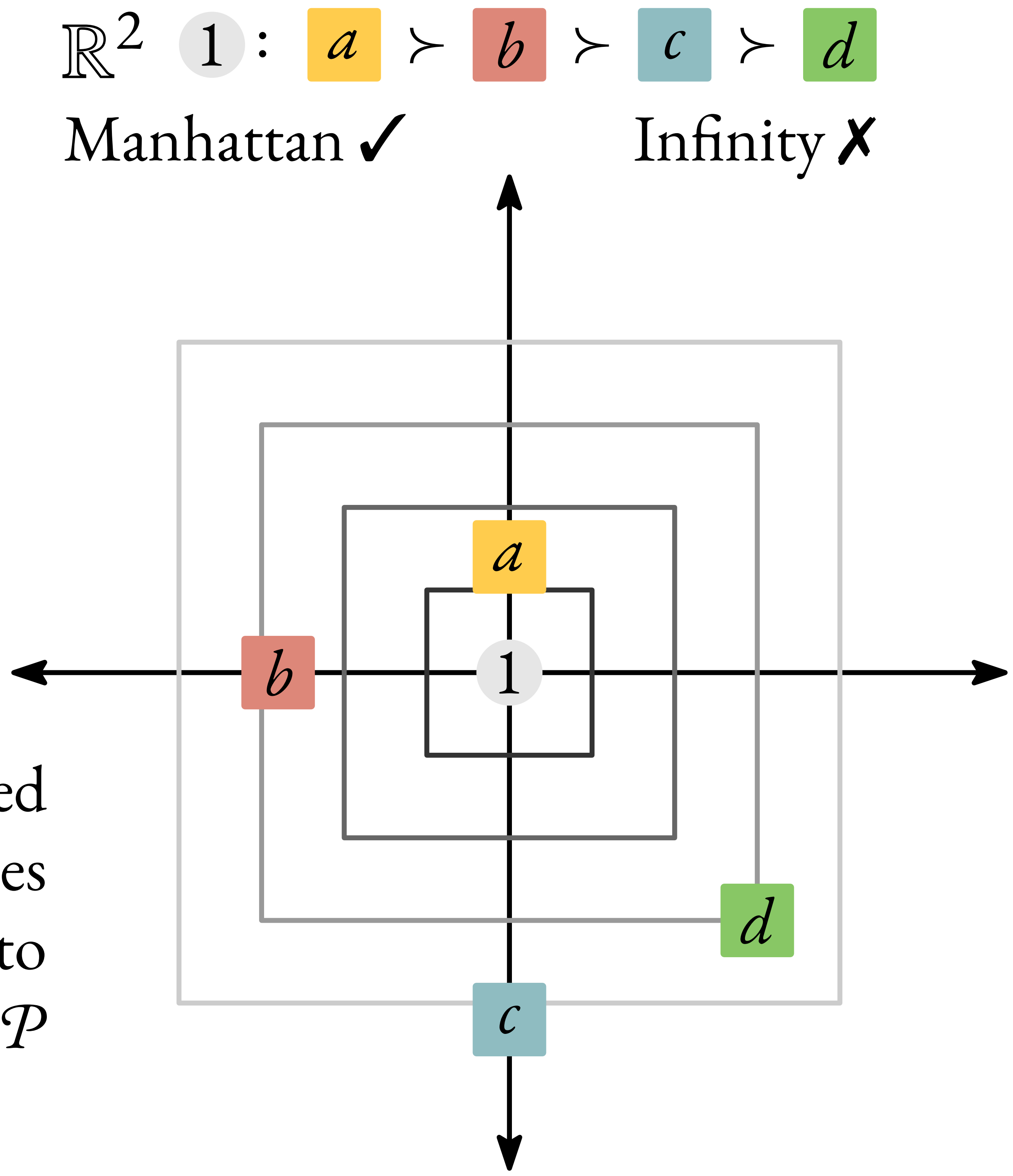

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Given the number of voters n and the number of alternatives m, for which dimensions d and norms $\|\cdot\|$ are rank-preserving embeddings guaranteed to exist?

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Rank embeddability under *p*-norms: $d \geq n, p > 1$

Construction: Alternative-Rank Embedding

- Choose $c \in \mathbb{R}$
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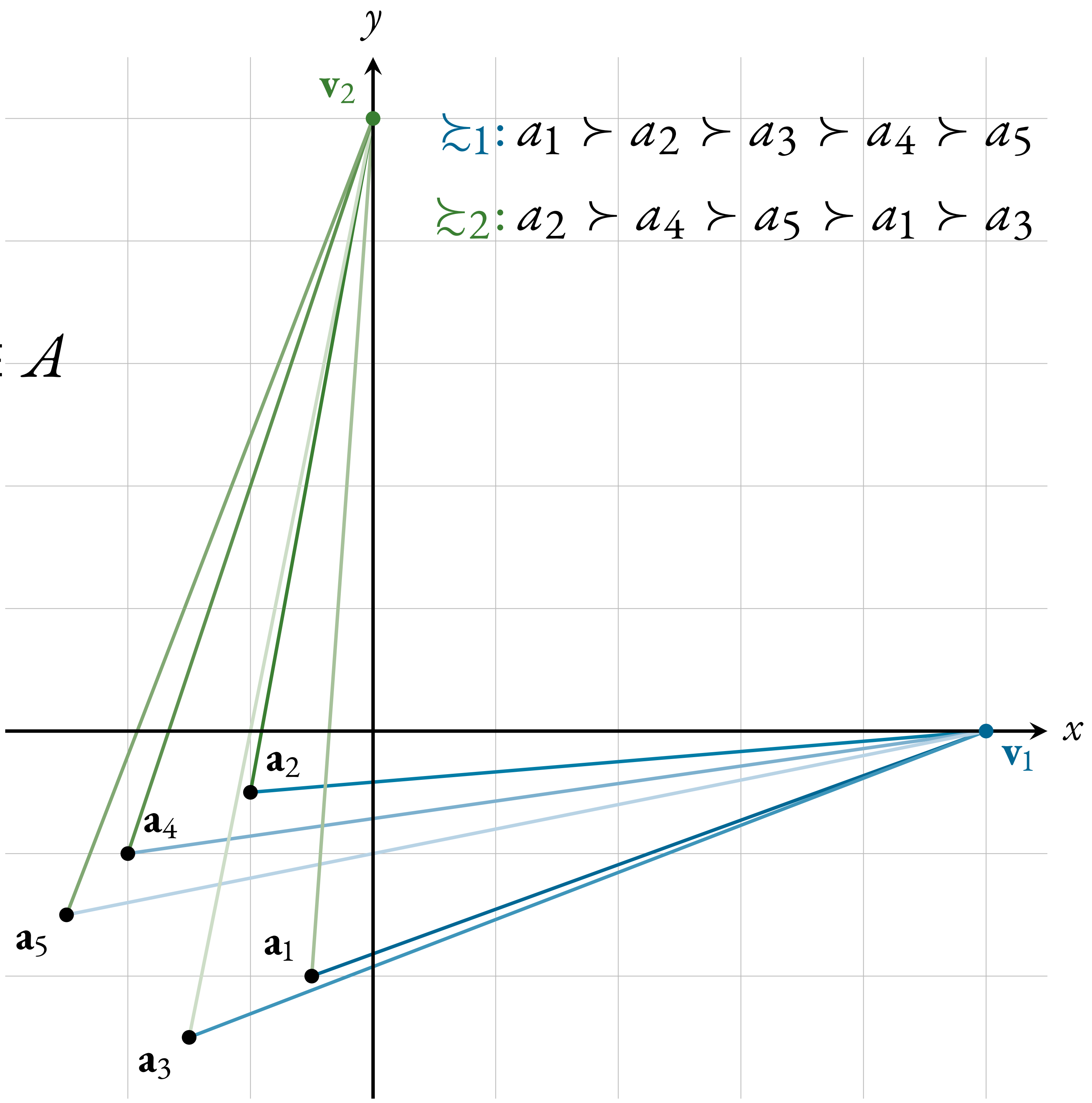

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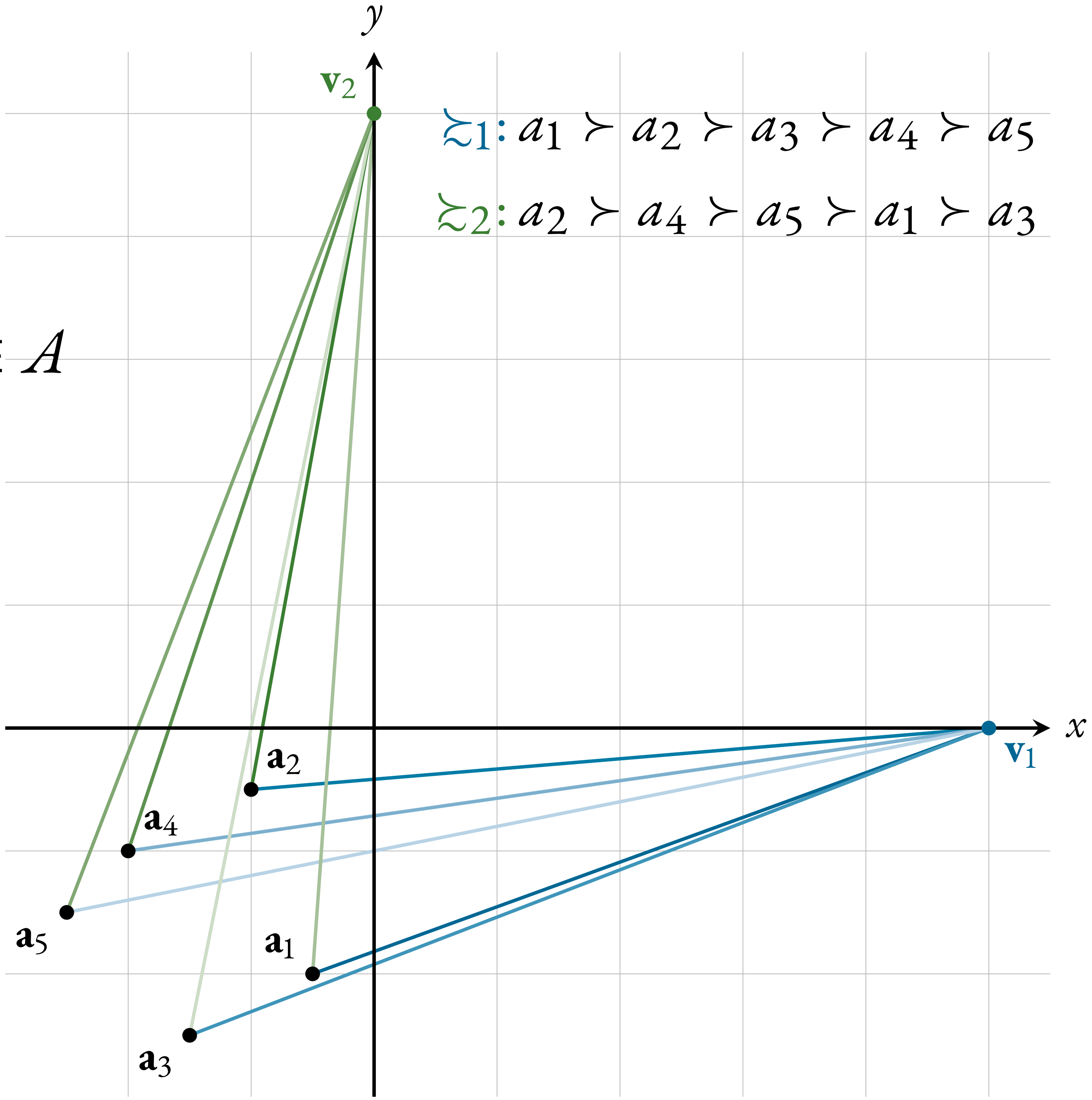
Proof intuition

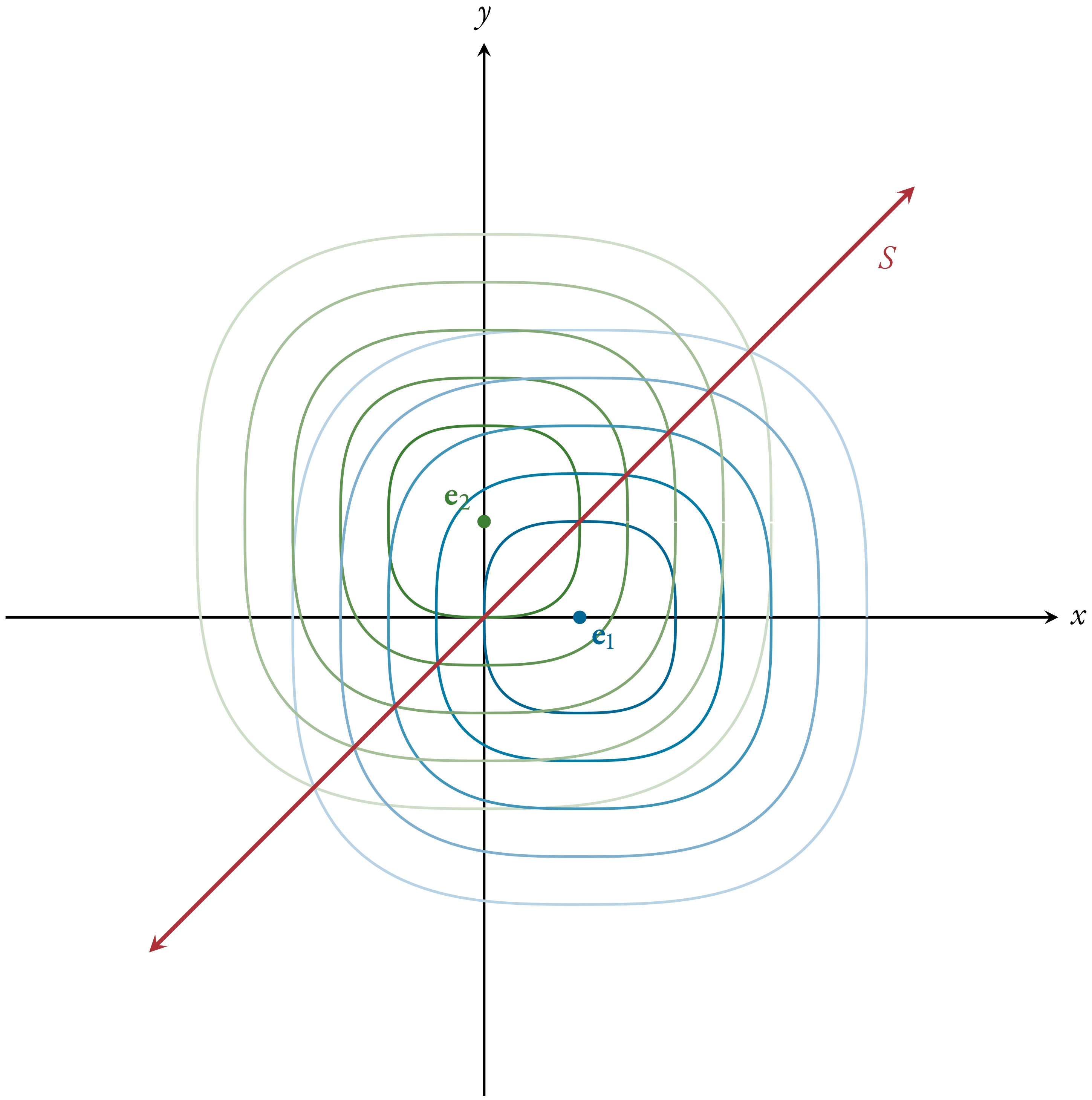
- Show that we can choose c sufficiently large to ensure rank preservation
- $1 < p < \infty$: c must satisfy $(c+2)^p - (c+1)^p > (n-1)(n^p - 1)$ $\longrightarrow$ always exists for fixed n, p
- $p = \infty$: $c = m$ works



Rank embeddability under p-norms: $d \geq m - 1, p > 1$

Proposition
For $1 < p < \infty$, $m > 1$, and $1 \leq i \neq j \leq m$,

$$S := \{\mathbf{x} \in \mathbb{R}^m \mid \|\mathbf{x} - \mathbf{e}_i\|_p = \|\mathbf{x} - \mathbf{e}_j\|_p\}$$
$$= \{\mathbf{x} \in \mathbb{R}^m \mid x_i = x_j\}\ .$$



Proof intuition
Definition + properties of p-norms

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- Alternatives: $\mathbf{a}_i = \mathbf{e}_i$ for $a_i \in A$
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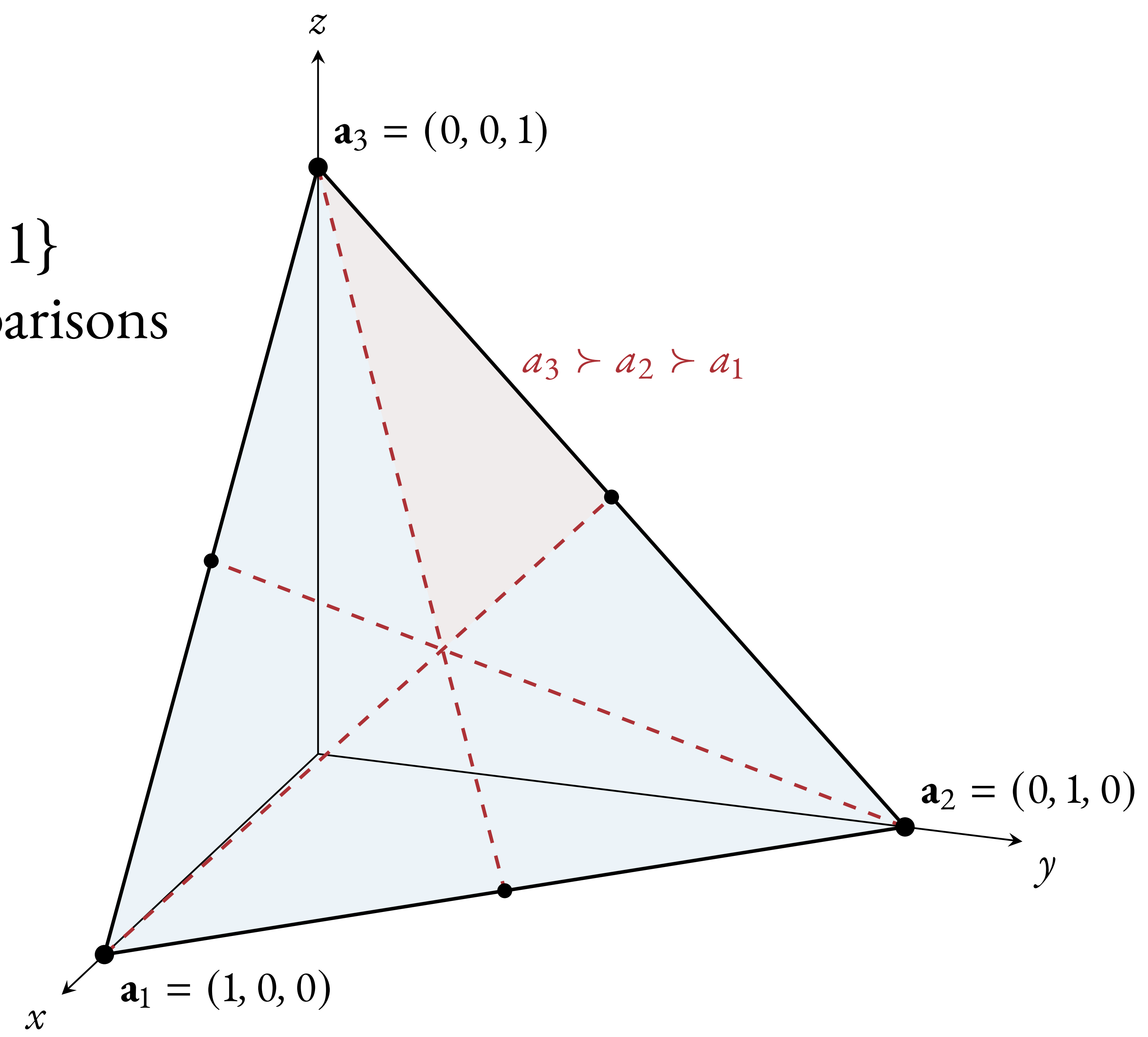
- Voters: Coordinates reflect pairwise comparisons



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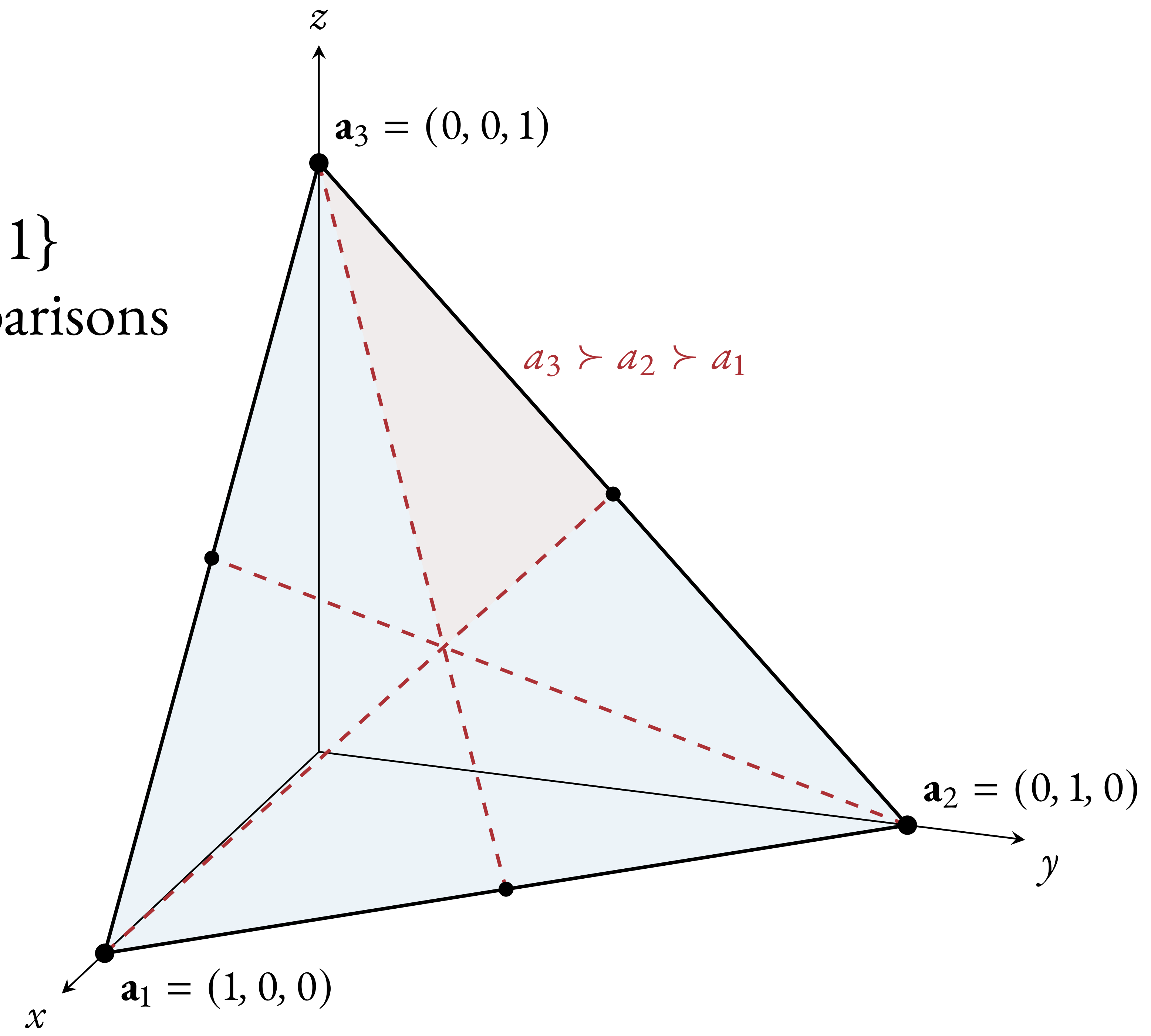
Proposition about hyperplanes ($1 < p < \infty$)
+ linearity of $\mathcal{S} \cap \mathcal{P}$ ($p = \infty$)



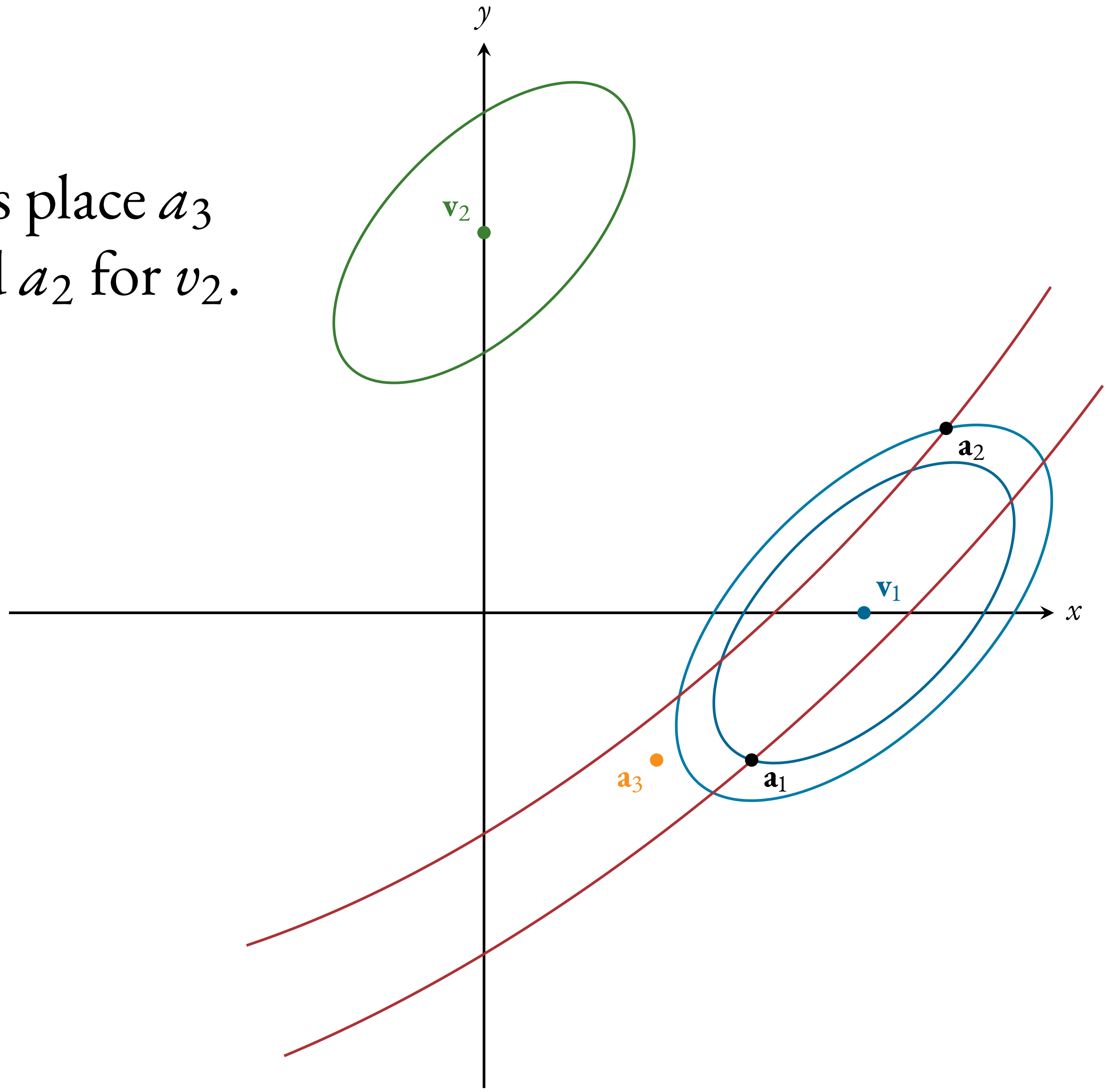
Rank embeddability under arbitrary norms: $d = n = 2$

Lemma

With two v_1, v_2 and a_1, a_2 placed, we can always place a_3 such that it ranks last for v_1 and between a_1 and a_2 for v_2.

Proof intuition

Fundamental geometry + properties of norms



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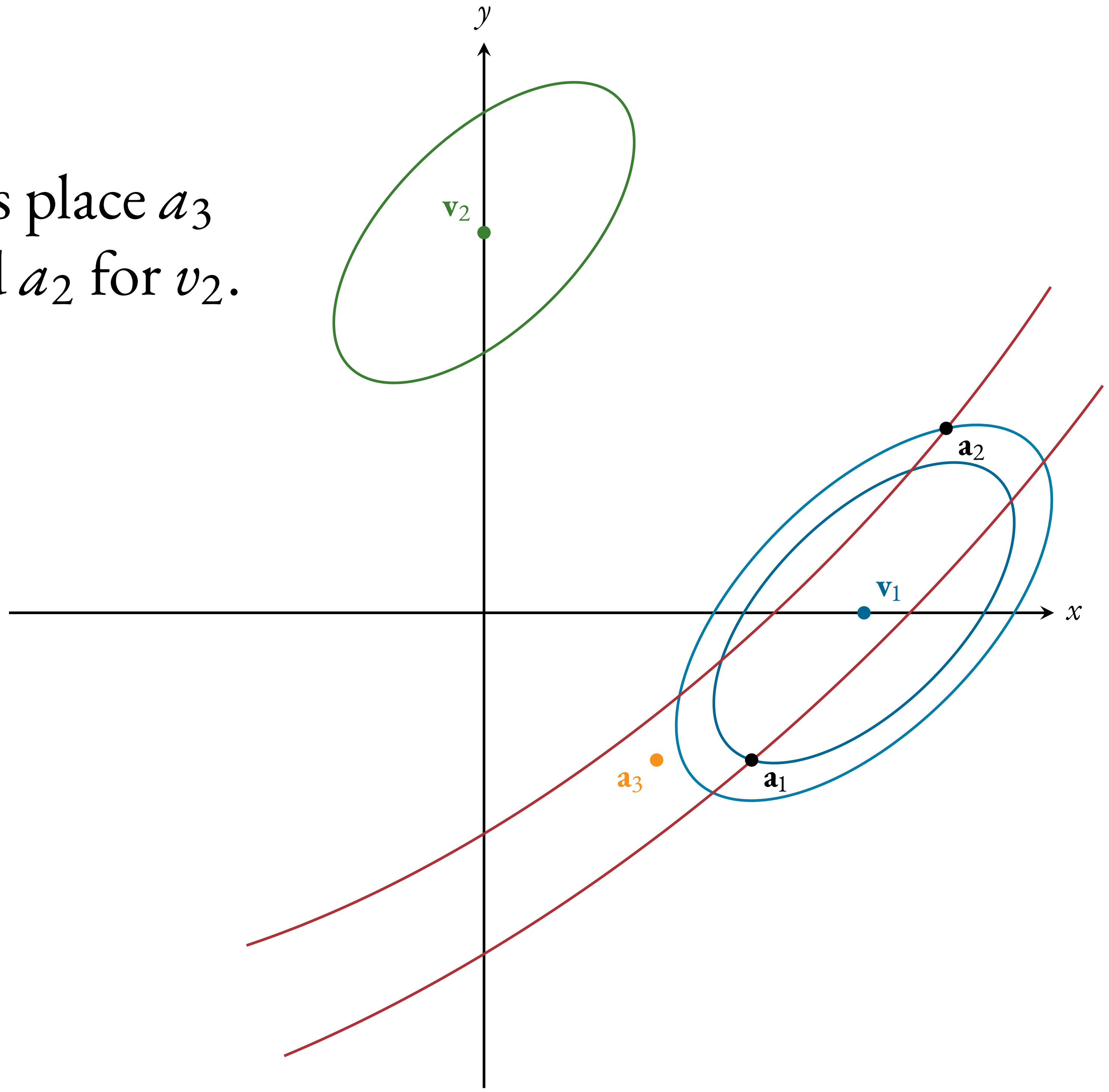
Proof intuition

Fundamental geometry + properties of norms

Proof intuition (Theorem 2)

Induction on the hypothesis that for any v_1, v_2 placed at $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$, there exist $\mathbf{a}_1, \ldots, \mathbf{a}_m$ s.t. the preference orderings are preserved and $\mathbf{v}_2 \in \overline{B(\mathbf{v}_1, \max\{\|\mathbf{v}_1 - \mathbf{a}_i\| \mid 1 \leq i \leq m\})}^c$.



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Thank you! Questions, comments, suggestions?

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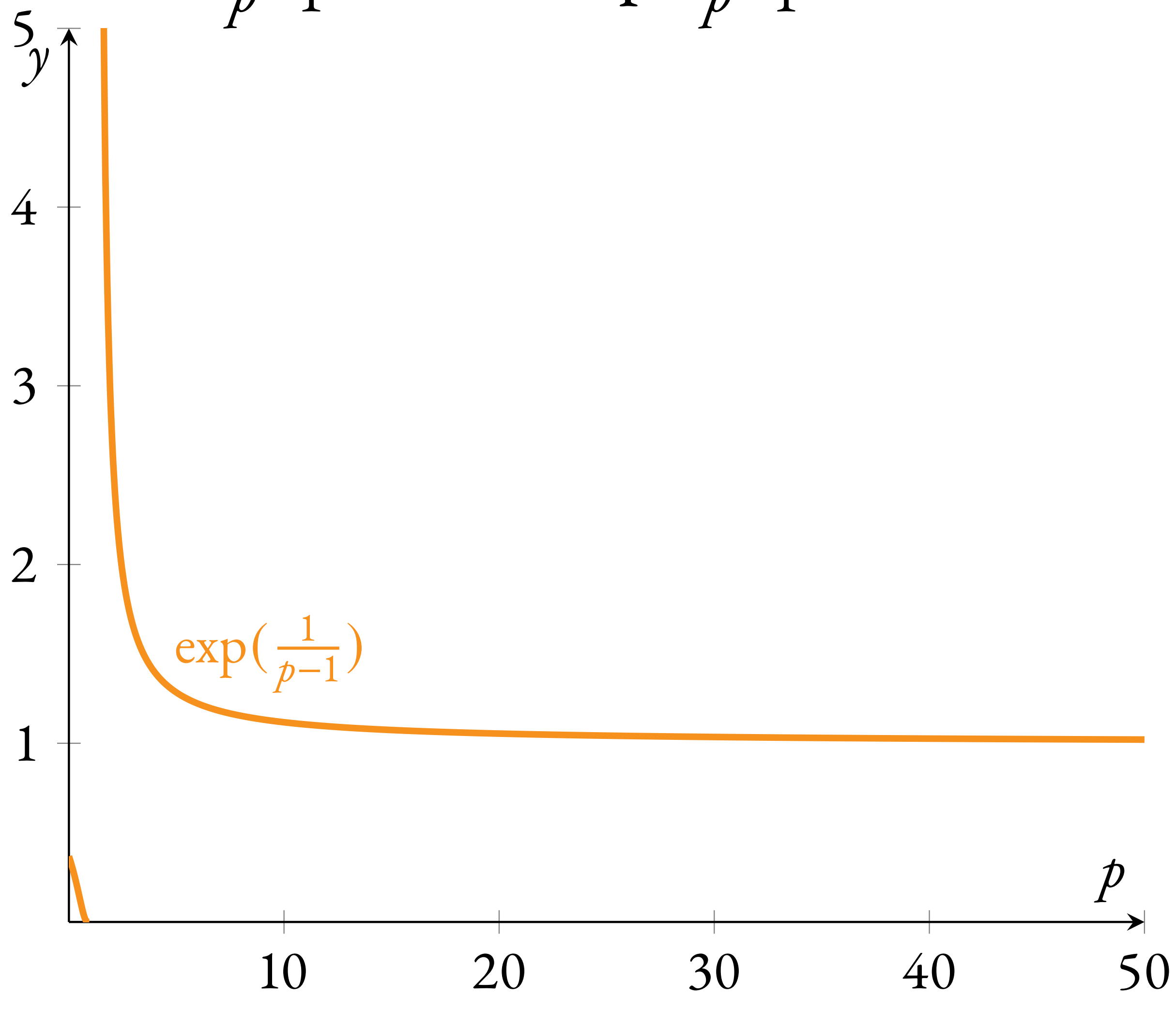
Appendix: Rank embeddability under p-norms: $d \geq n, p = 1$

Why do alternative-rank embeddings fail for the Manhattan norm?

Let $c(x) := \inf\{c \mid (c+2)^{1+1/x} - (c+1)^{1+1/x} > (n-1)(n^{1+1/x} - 1)\}$, with $n \in \mathbb{N}$ and $p > 1$. Then $c(\frac{1}{p-1}) \in \Theta(\exp(\frac{1}{p-1}))$.



Proof intuition

Mean value theorem + clever function definitions

Fix: Max-Rank Embedding (Chen et al. 2022)

- Voters: $\mathbf{v}_i = m\mathbf{e}_i$
- Alternatives:

$$a_j^{(i)} = \begin{cases} \mathrm{rk}_{ij} - \mathrm{mk}_j & i = g_j \\ c + 2\mathrm{rk}_{ij} + \sum_{k=1}^{n}(\mathrm{rk}_{kj} - \mathrm{mk}_j) & i \neq g_j \end{cases}$$

  where $g_j = \arg\max_i \mathrm{rk}_{ij}$ and $\mathrm{mk}_j = \max_i \mathrm{rk}_{ij}$
- Gist: "very different construction"