Distributed Algorithms for Local Potential Problems

GRAN SASSO SCIENCE INSTITUTE

Francesco d'Amore

Joint work with A. Balliu, T. Boudier, D. Olivetti, G. Schmid, and J. Suomela

Helsinki Algorithms & Theory Days
29 August 2025

The result

Theorem

For any local potential problem Π*, there exists a randomized LOCAL algorithm that solves* Π *with high probability in time* $O(\log^6 n)$.

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For any local potential problem Π*, there exists a randomized LOCAL algorithm that solves* Π *with high probability in time* $O(\log^6 n)$.

derandomization [Ghaffari, Harris, and Kuhn, FOCS '18]

+

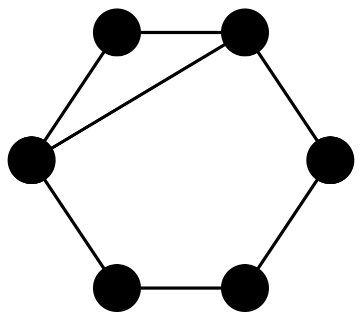
network decomposition [Ghaffari and Grunau, FOCS '24]

Corollary:

For any local potential problem Π*, there exists a deterministic LOCAL algorithm that solves* Π *in time* $O(\log^8 n \, \mathit{poly}(\log\log n))$.

Locally Optimal Cut (LOC)

Input: - graph $G = (V, E)$
- two colors red and green



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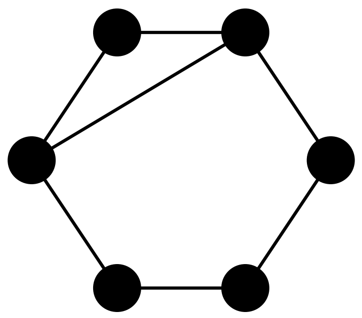

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- for each $v \in V$, at least $\geq \deg(v)/2$ *neighbors of different color* (w.r.t. v)

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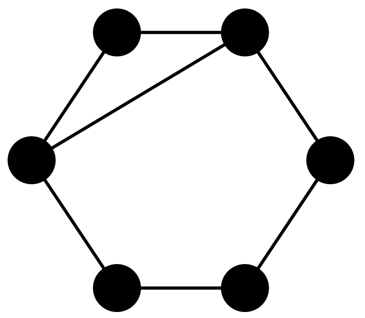

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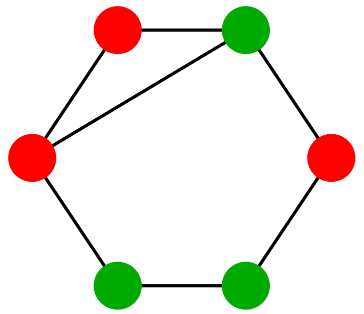

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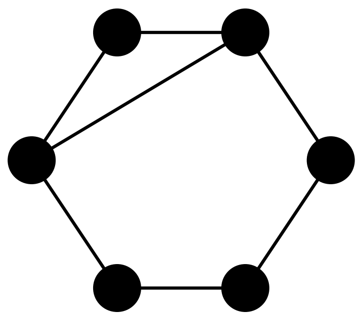

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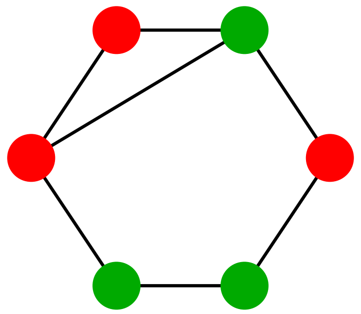

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2-*apx* of MAX-CUT (locally optimal)

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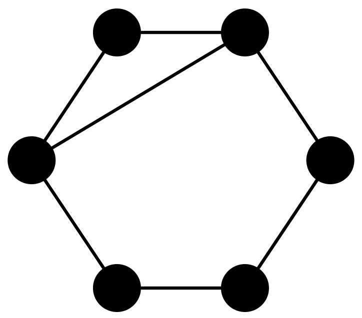

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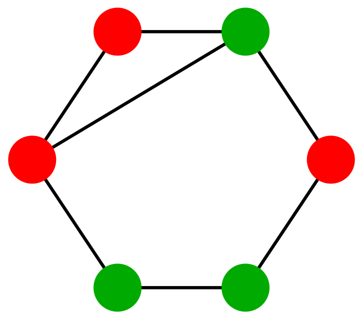

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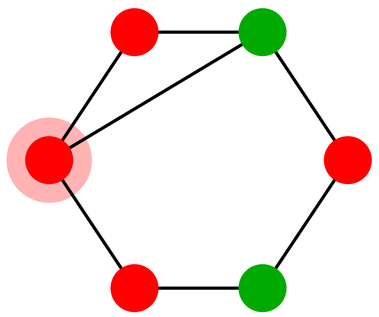

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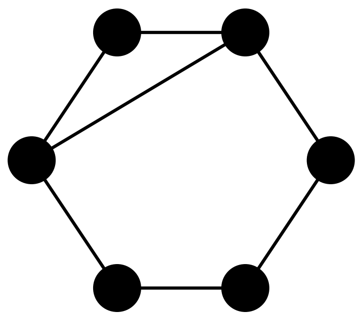

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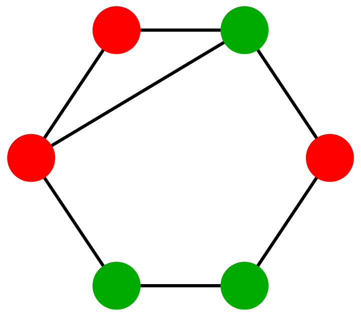

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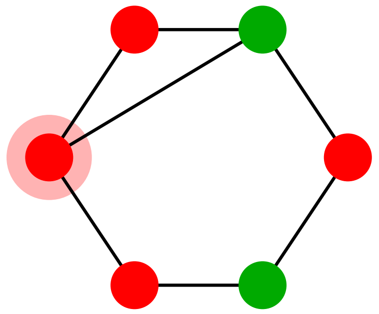
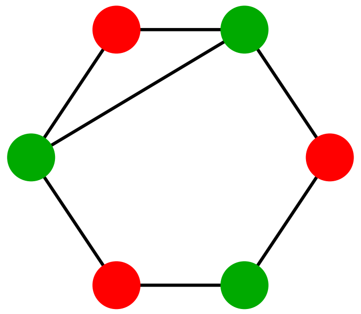


invalid solution

fixing procedure → *flip color of invalid node*



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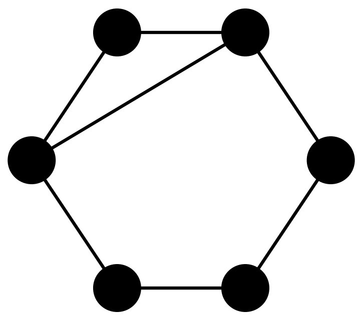

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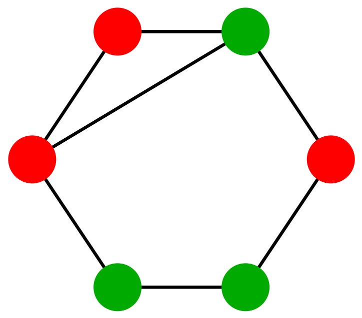

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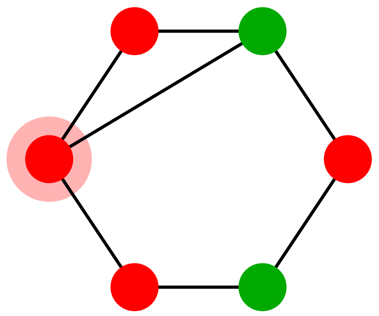

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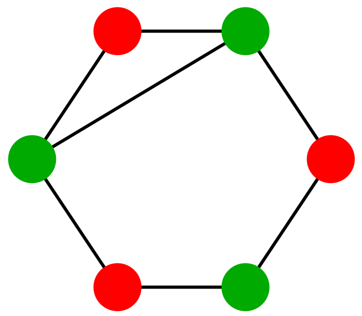
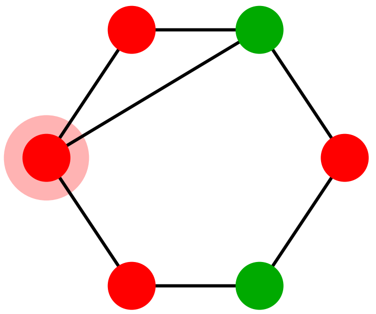


Question: can we *always* solve the problem?

LOC: fixing procedure

Fixing procedure: *pick an arbitrary invalid node and flip its color*



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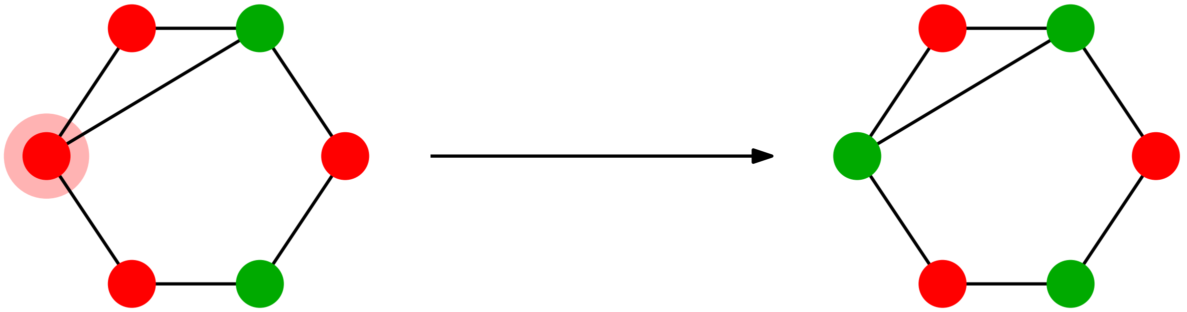

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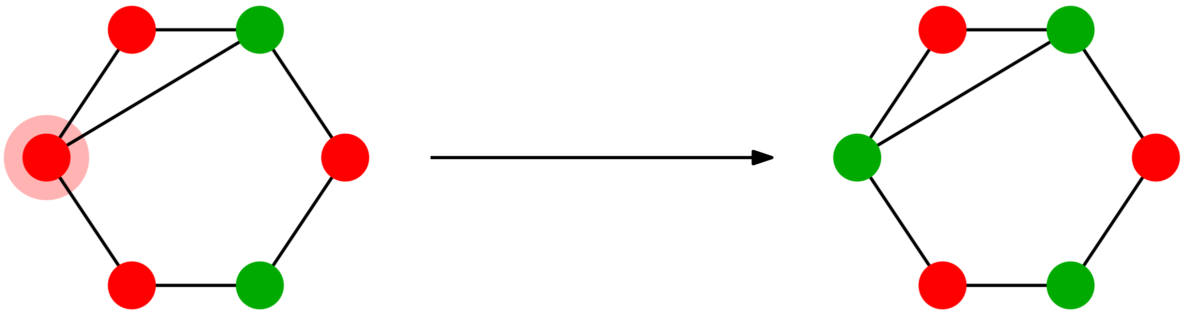

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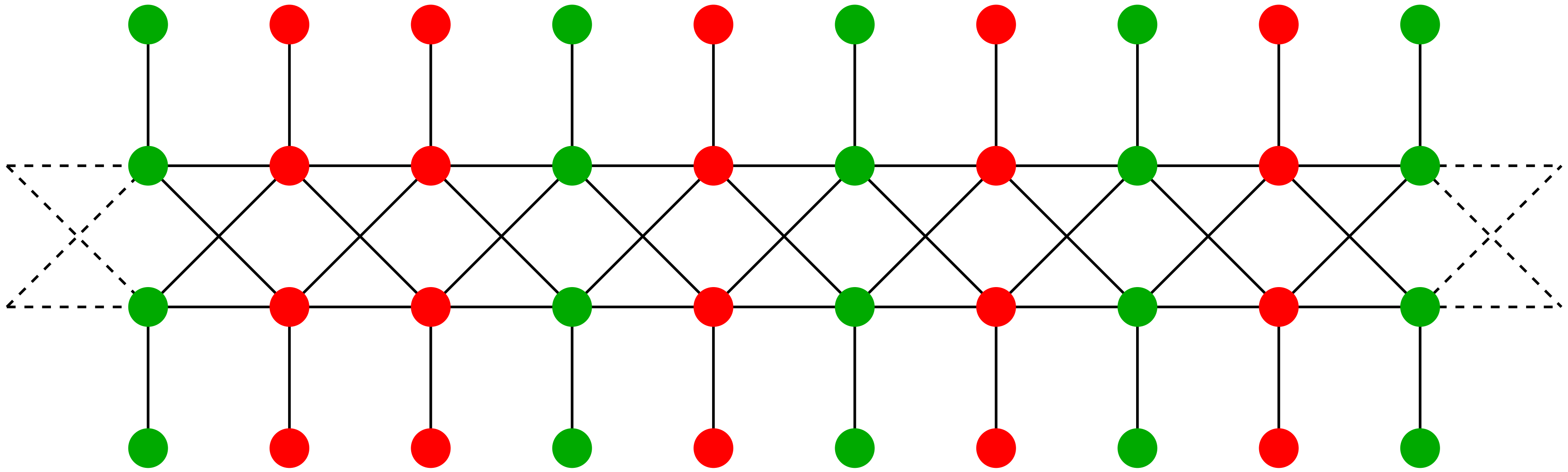

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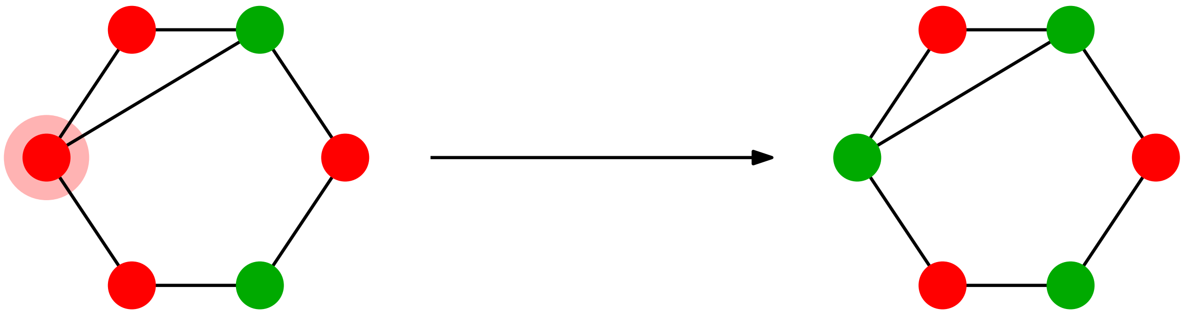

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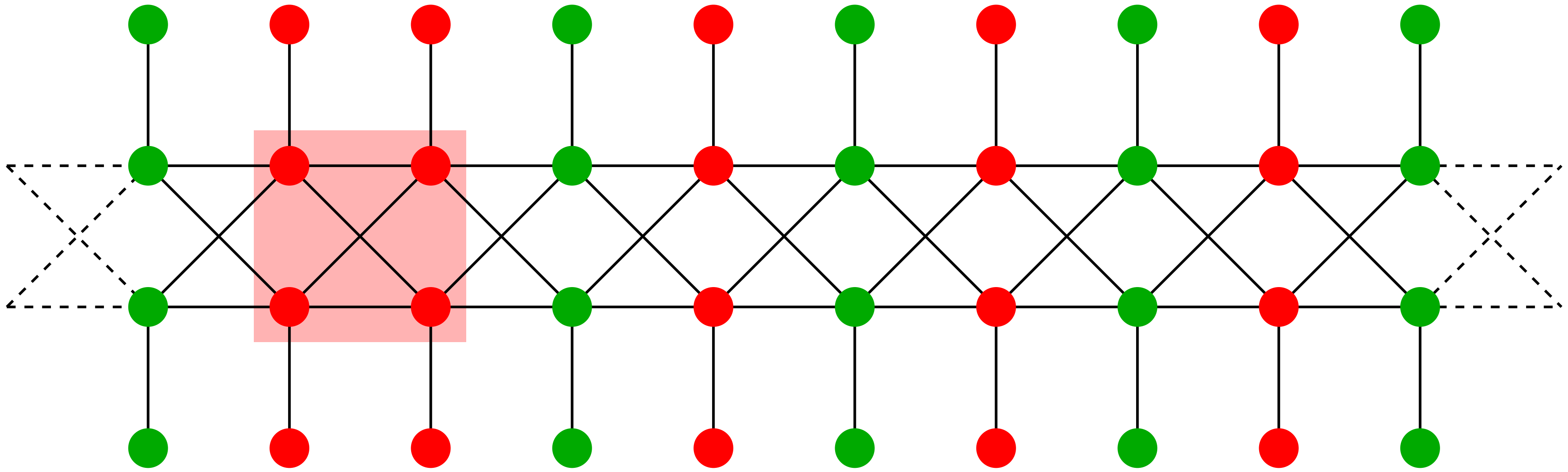

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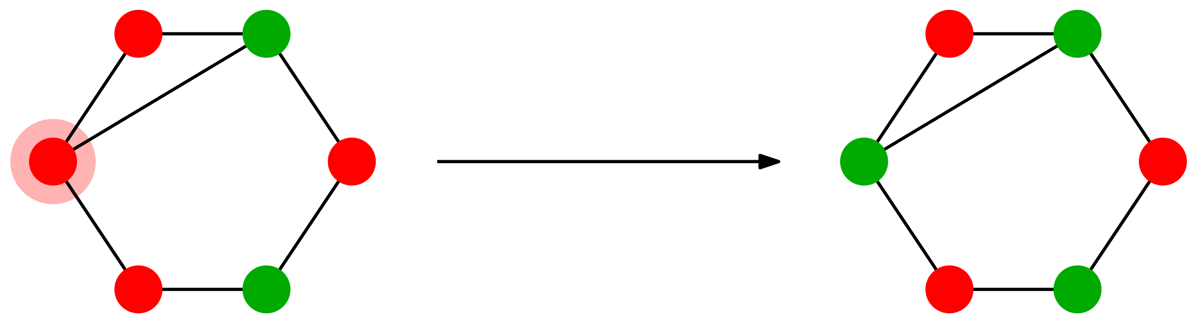

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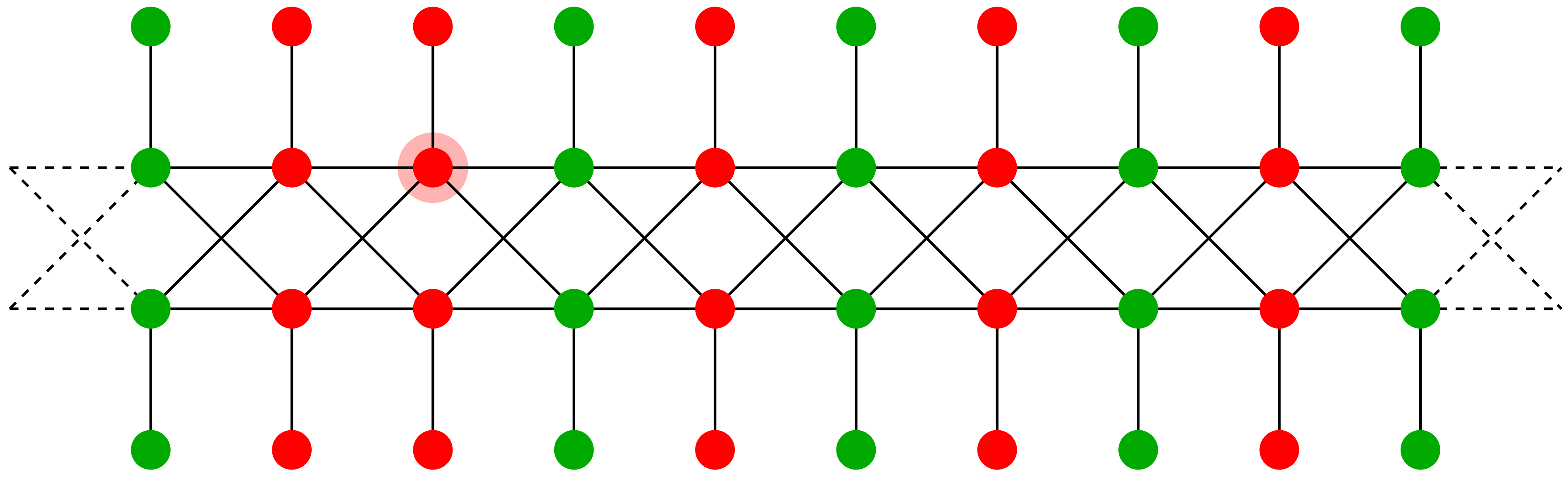

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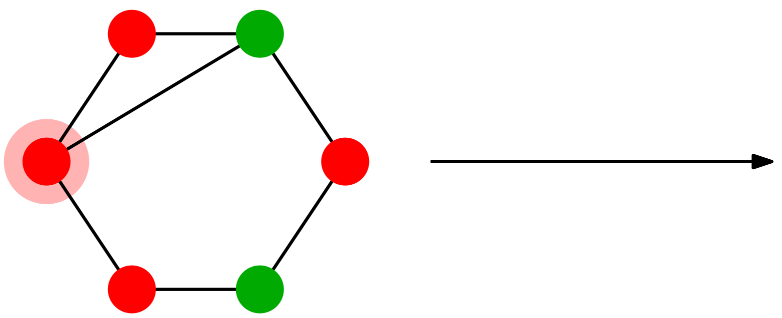

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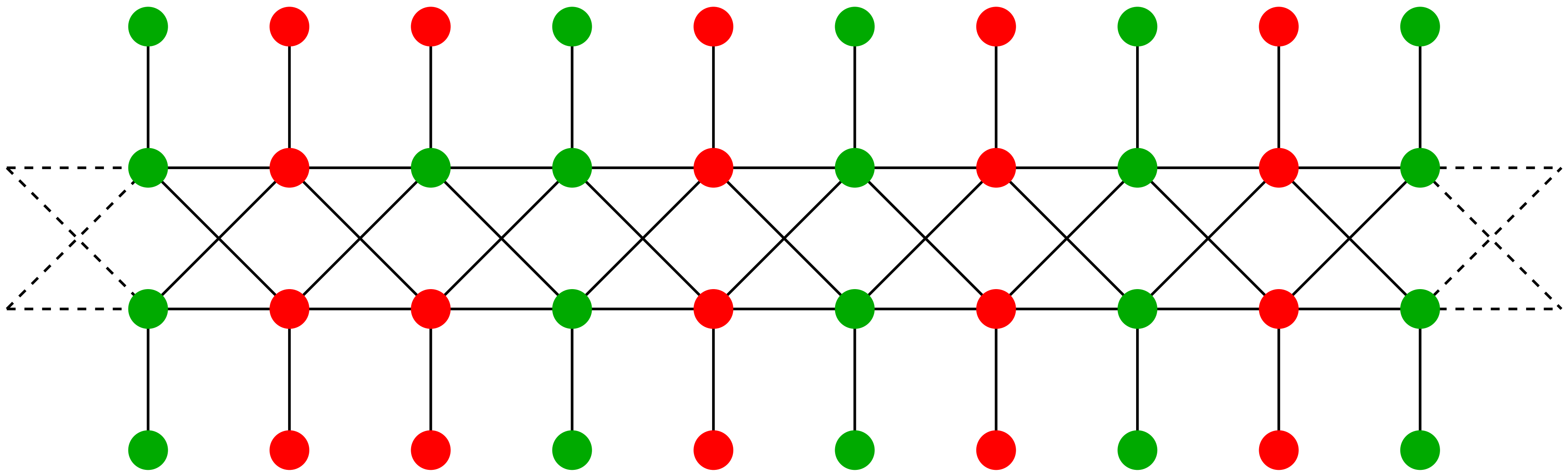

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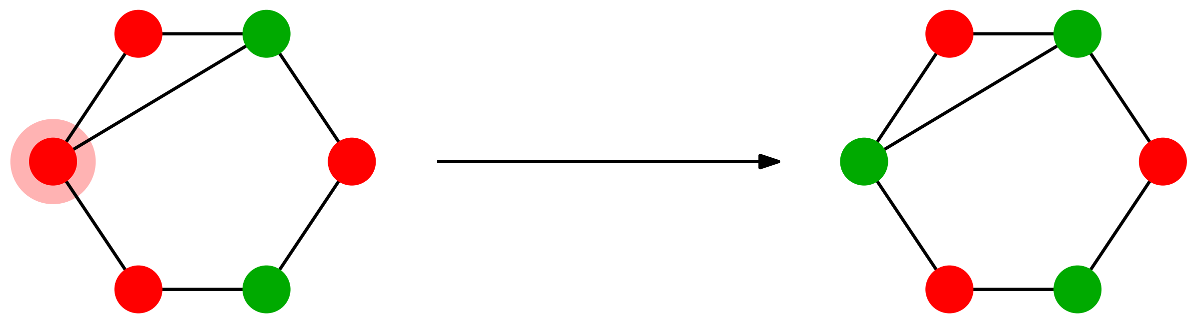

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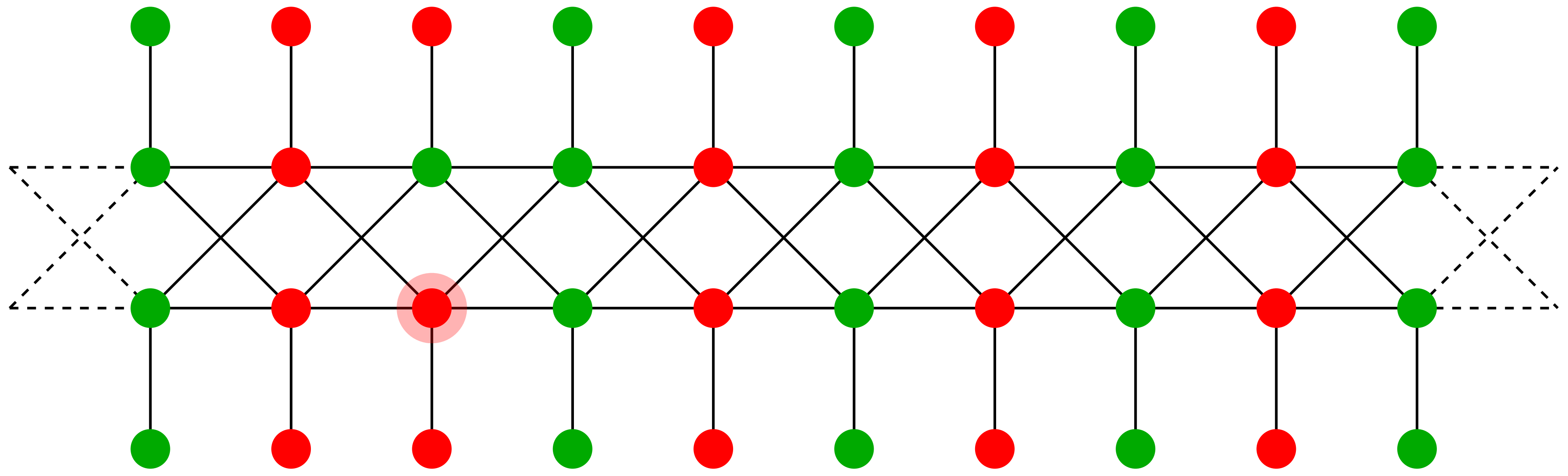

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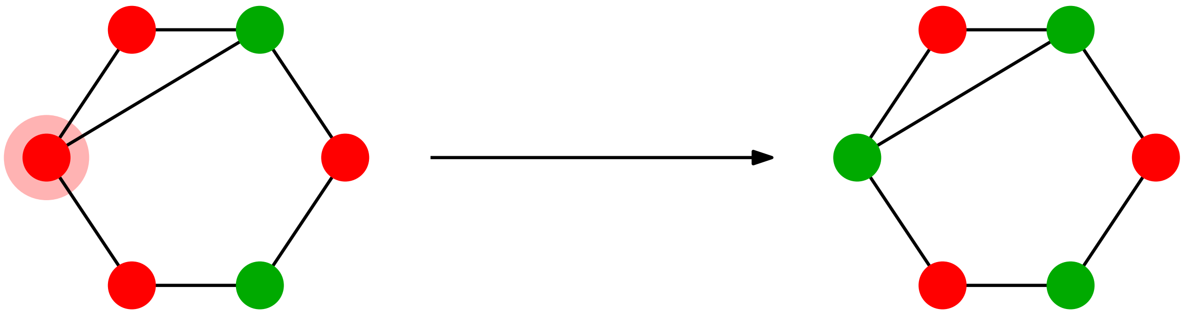

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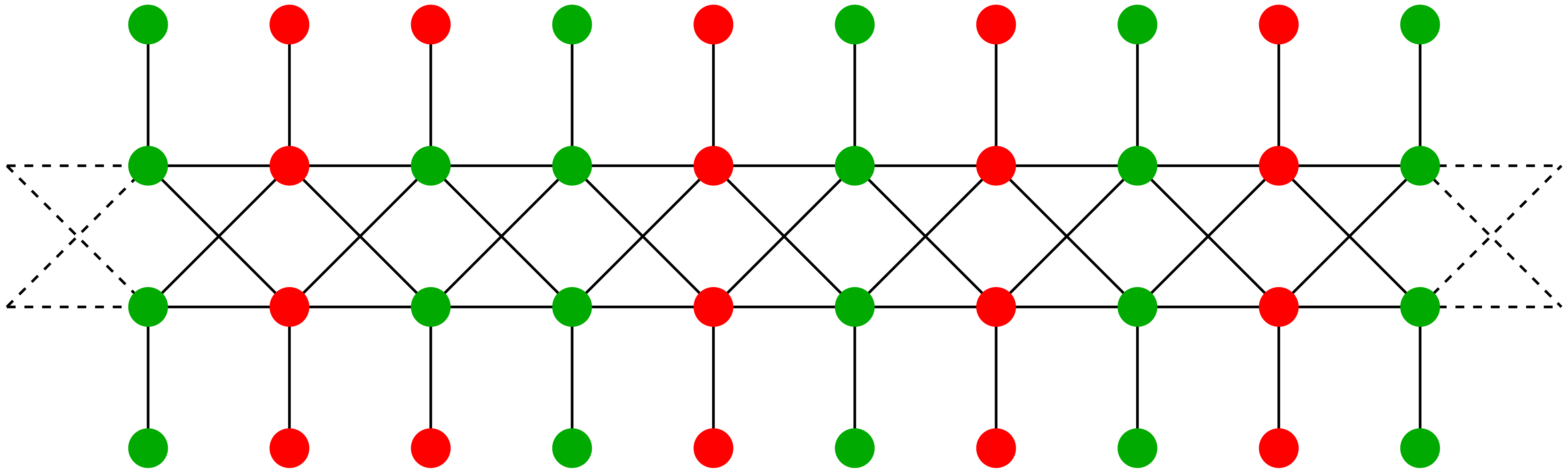

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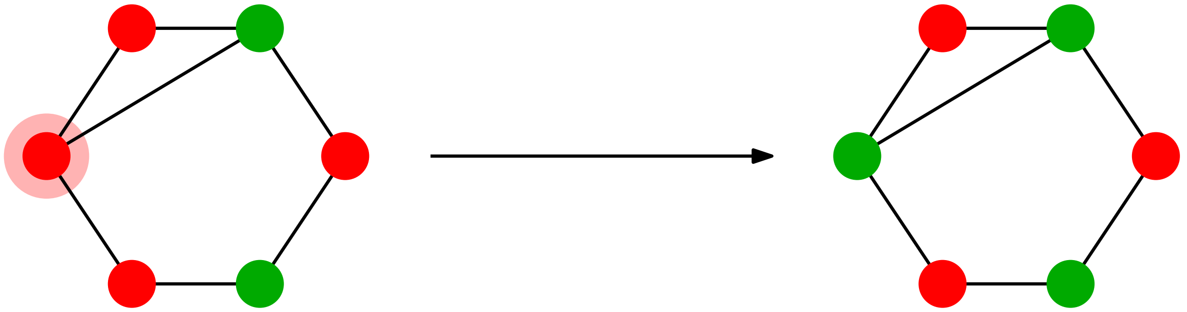

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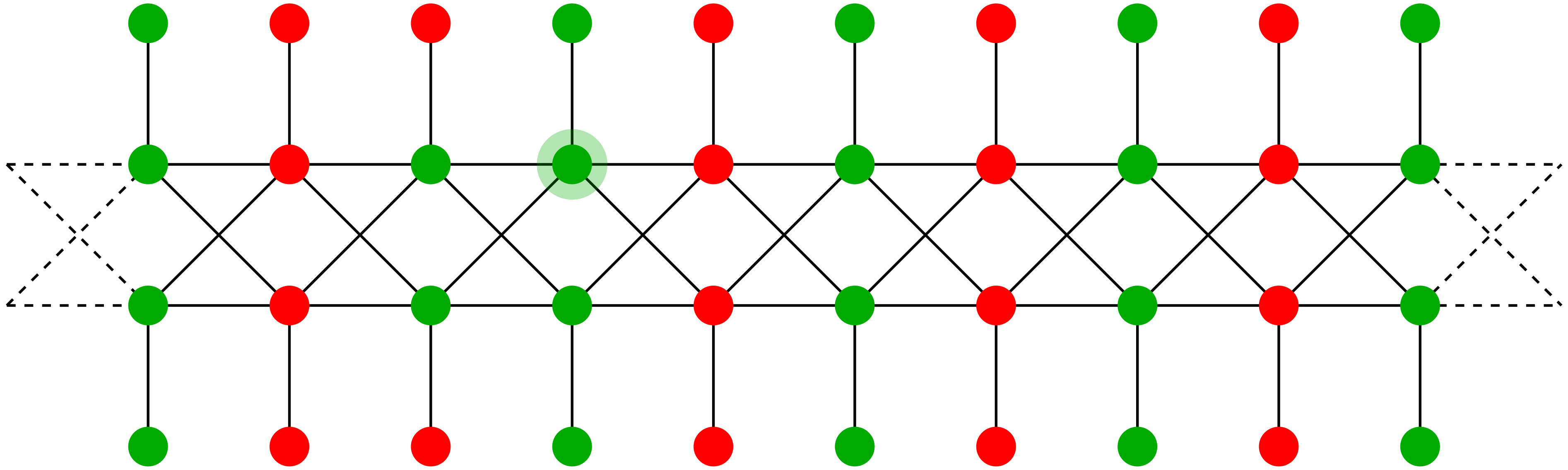

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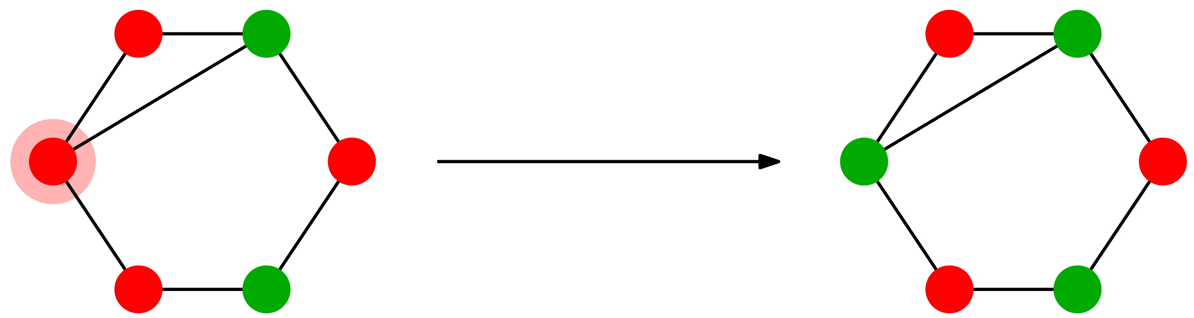

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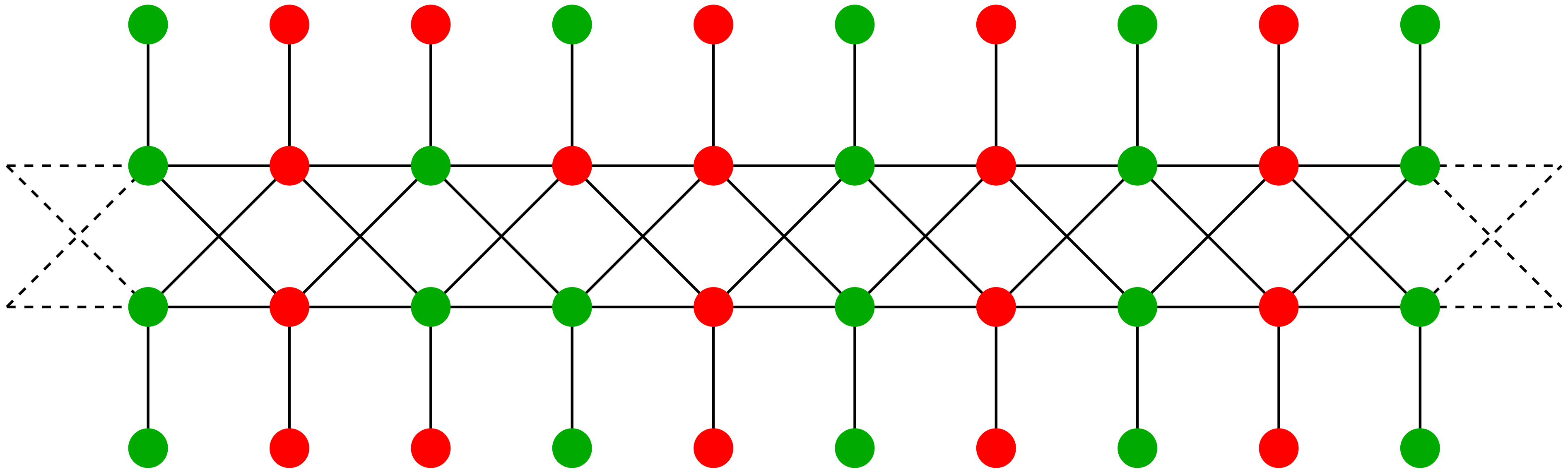

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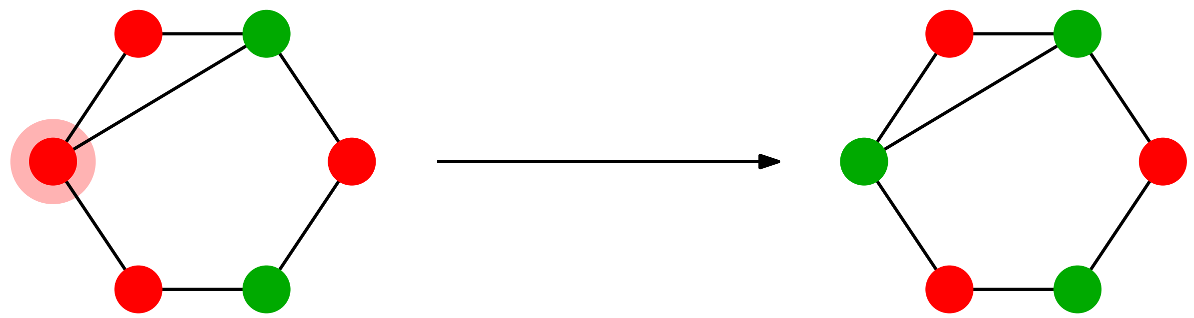

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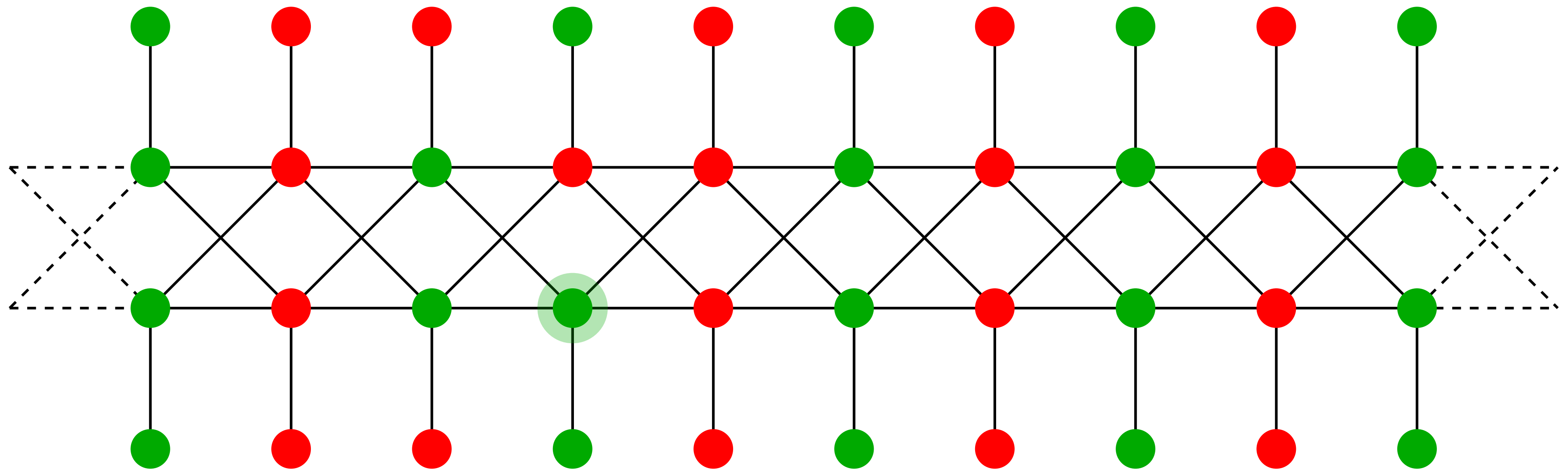

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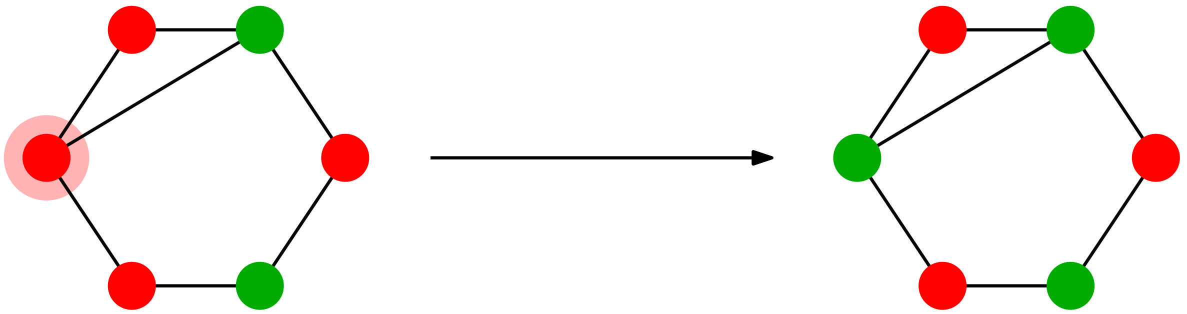

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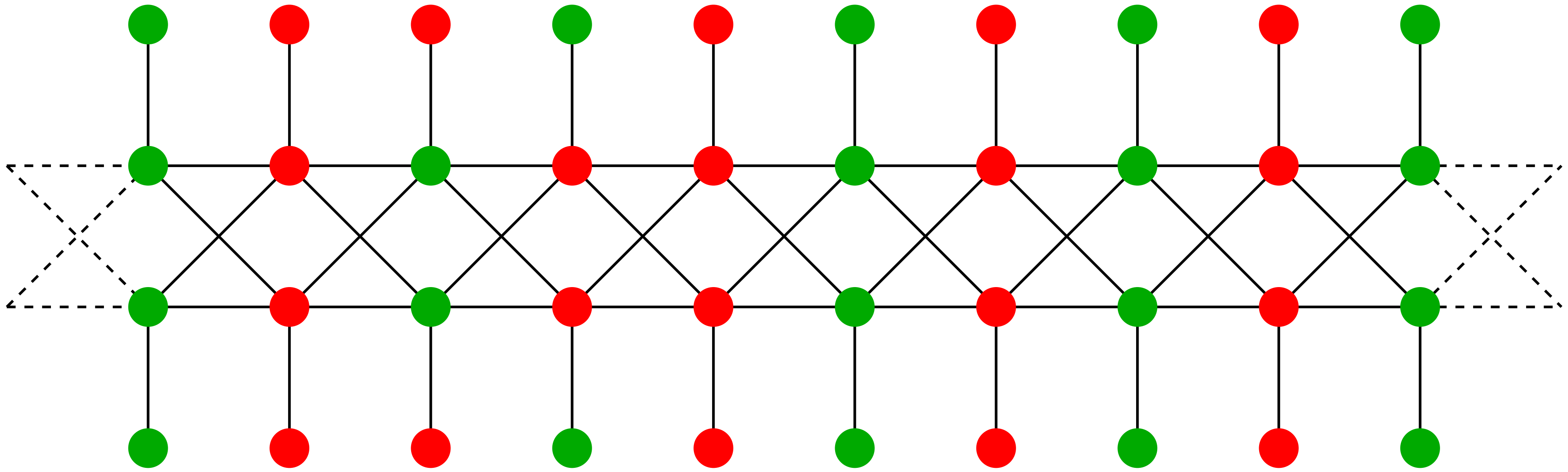

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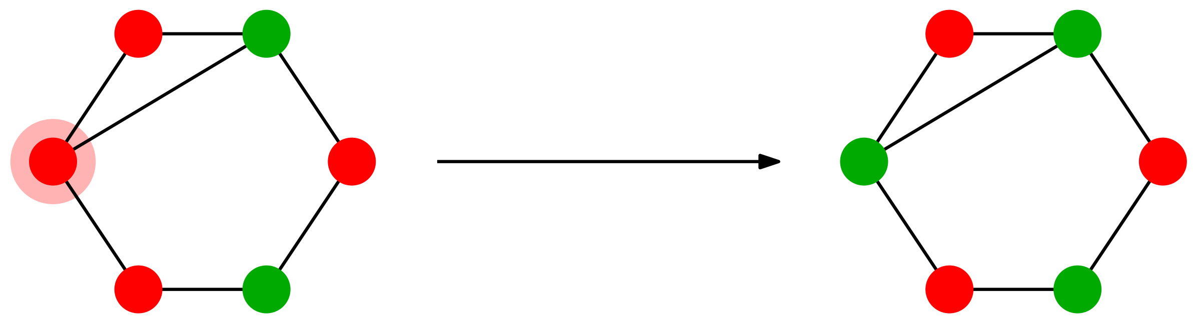

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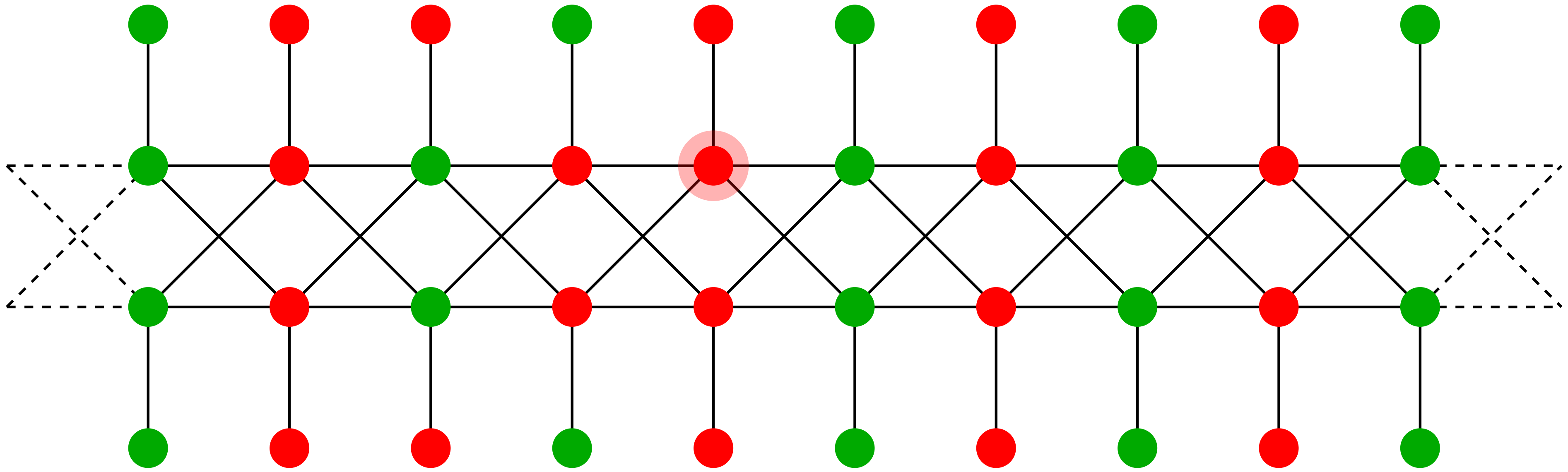

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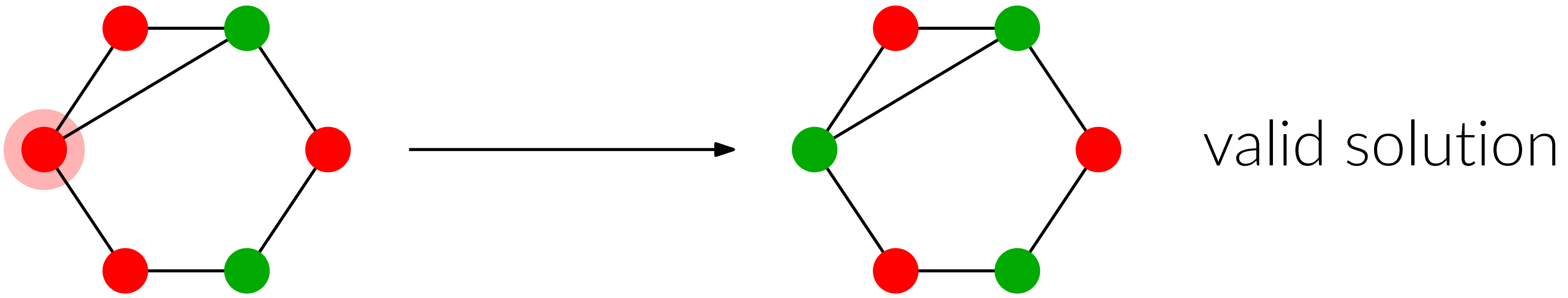

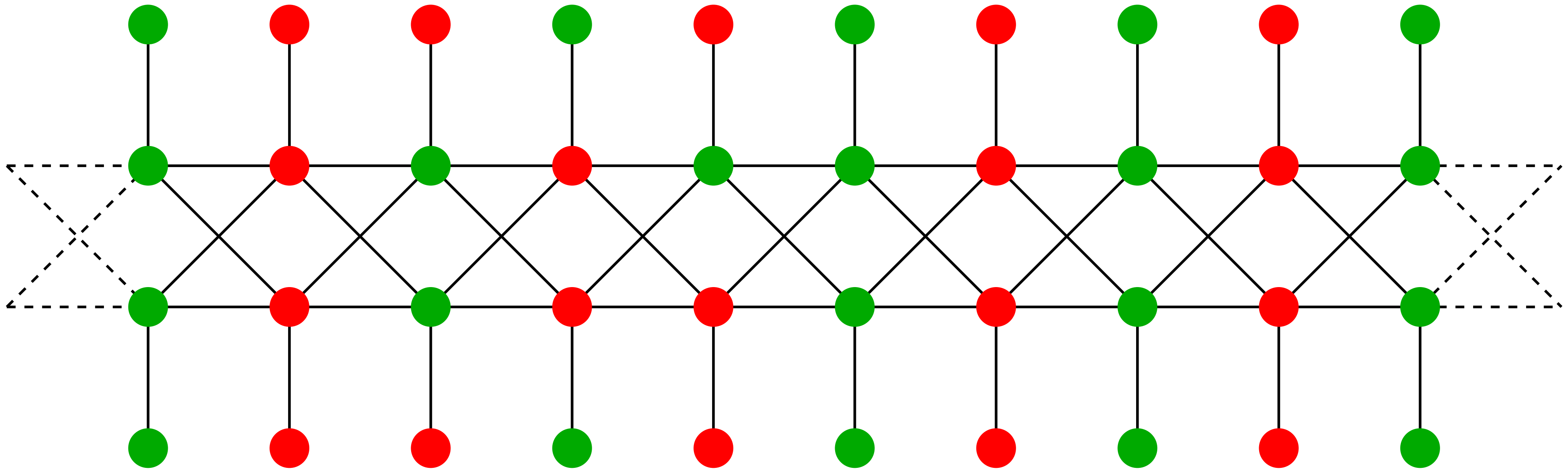

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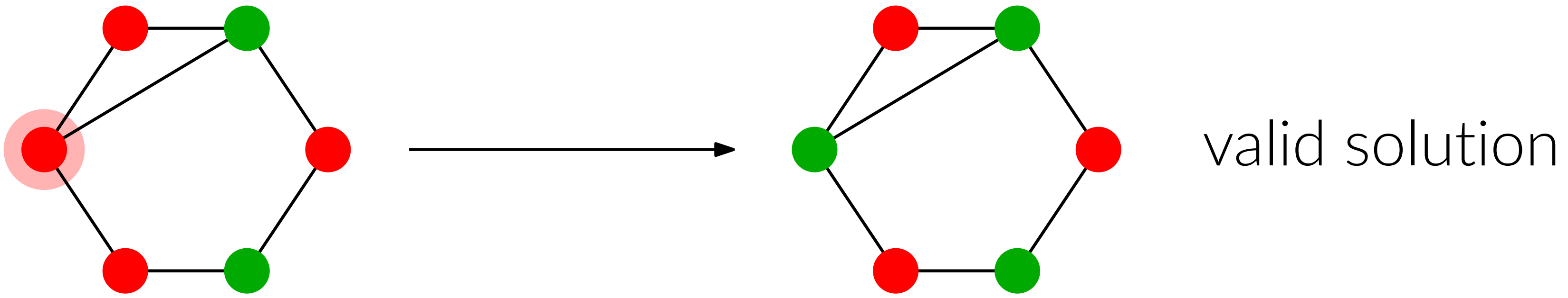

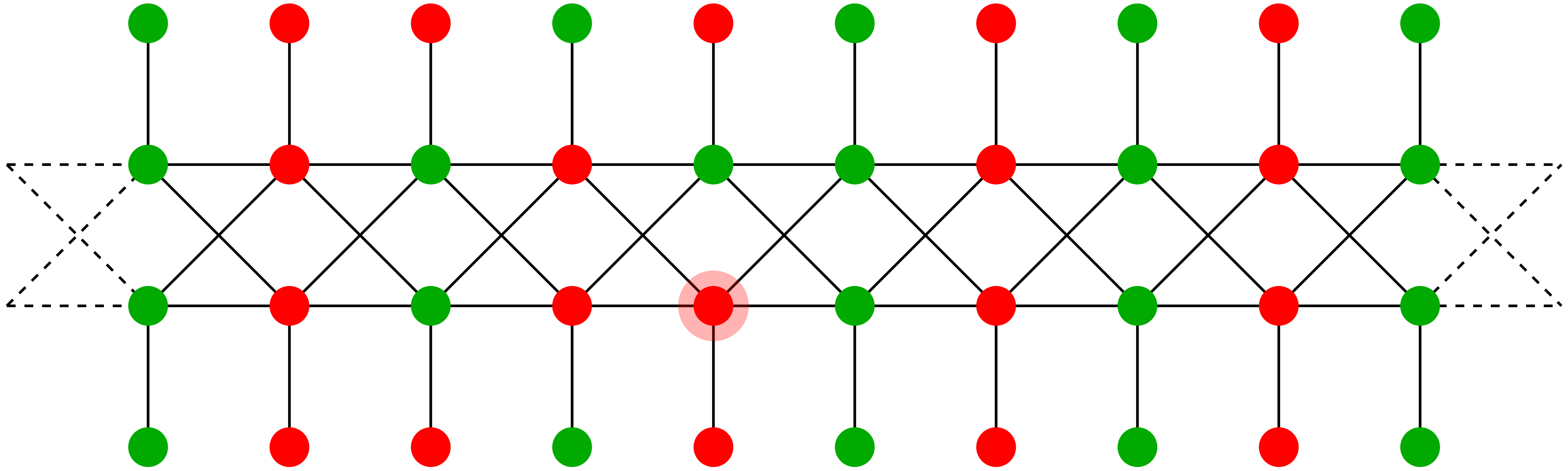

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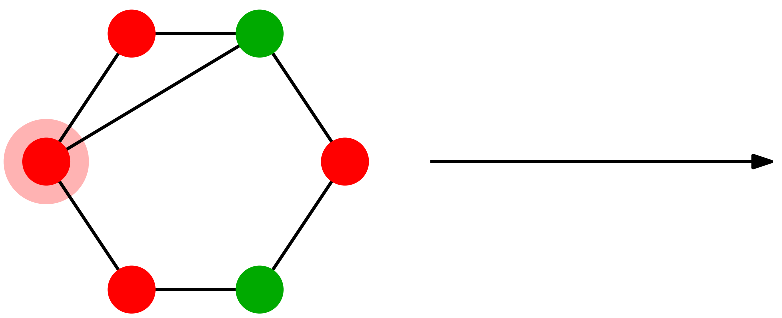

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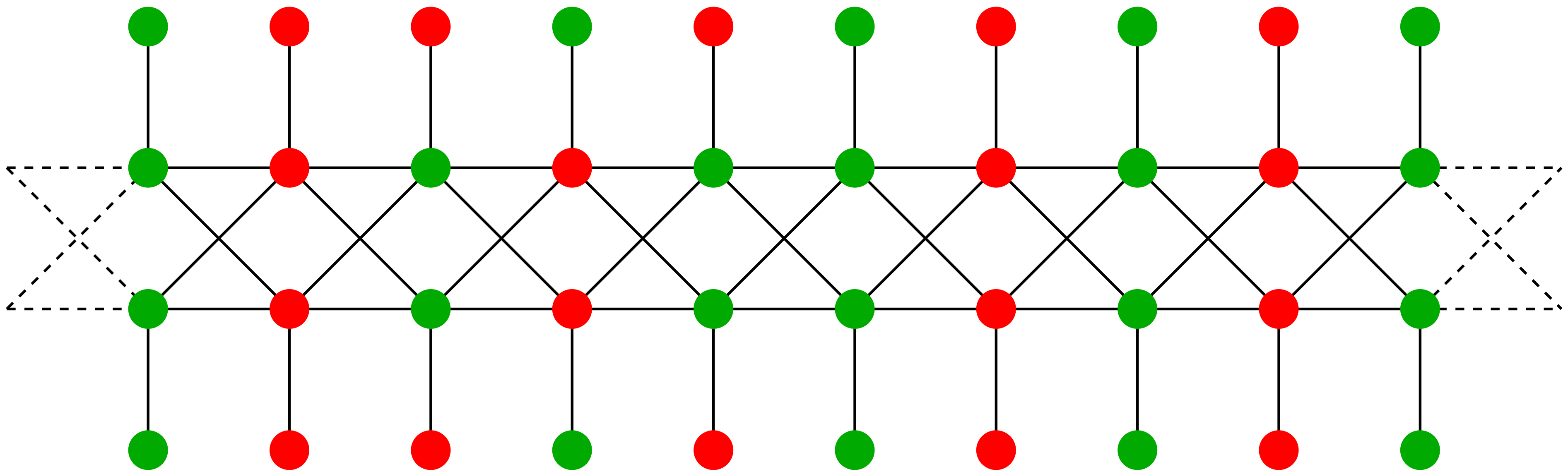

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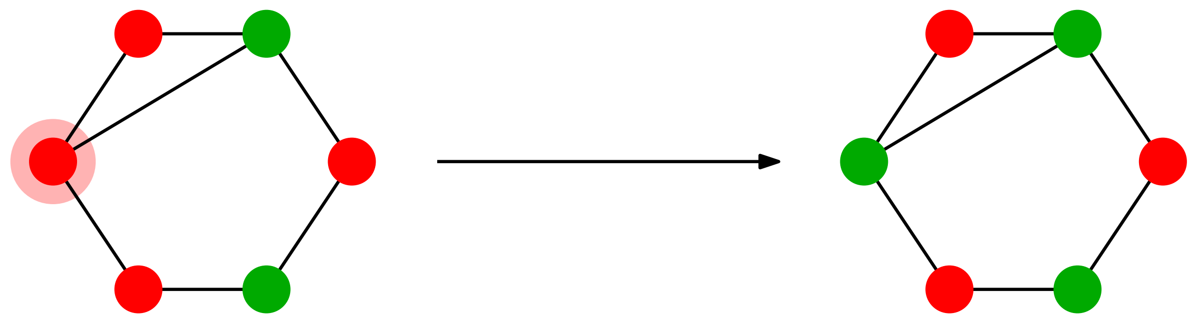

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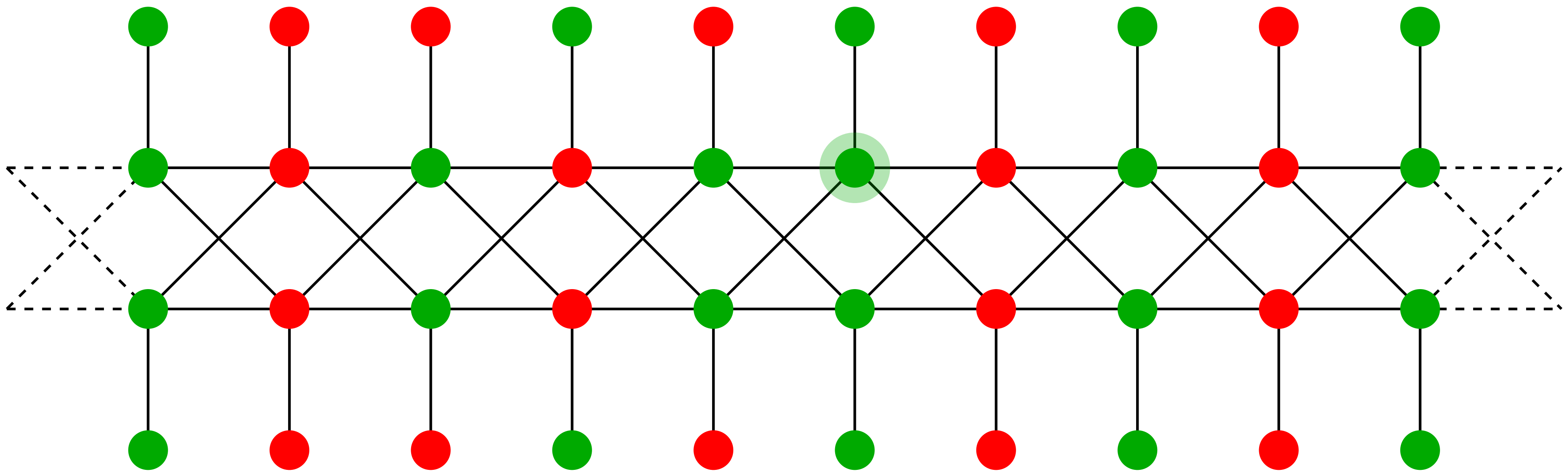

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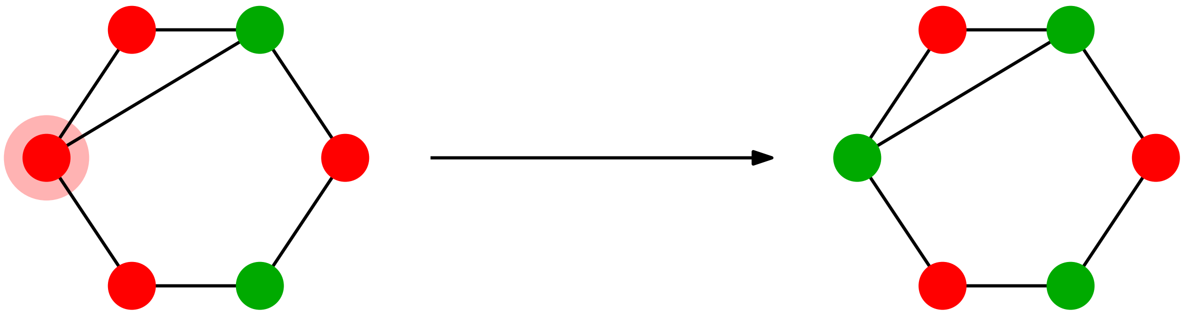

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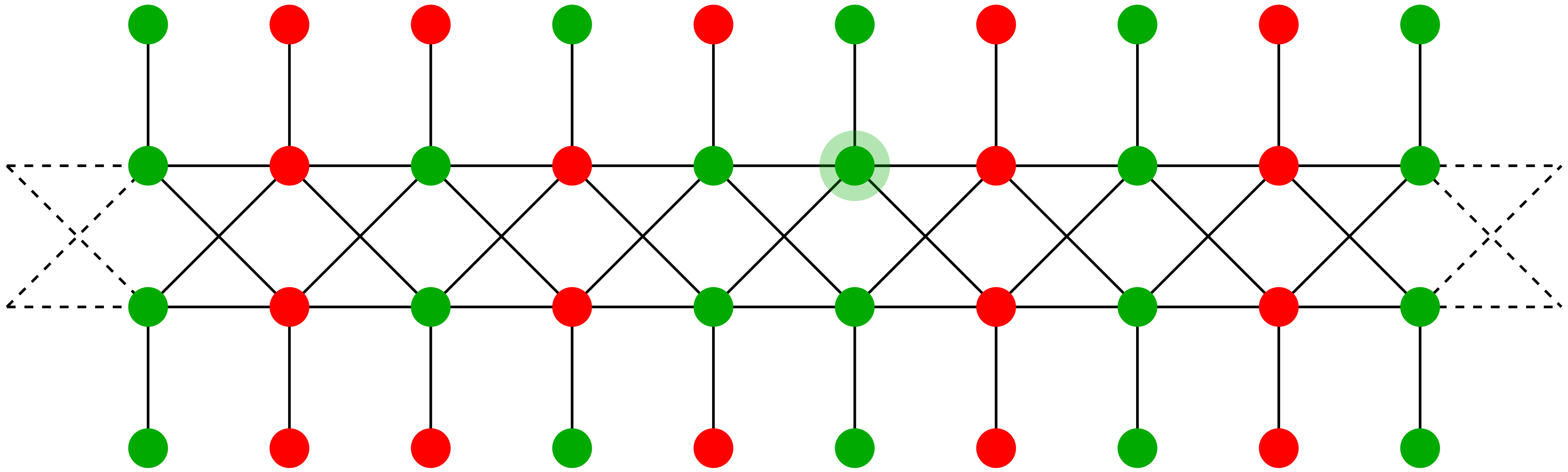
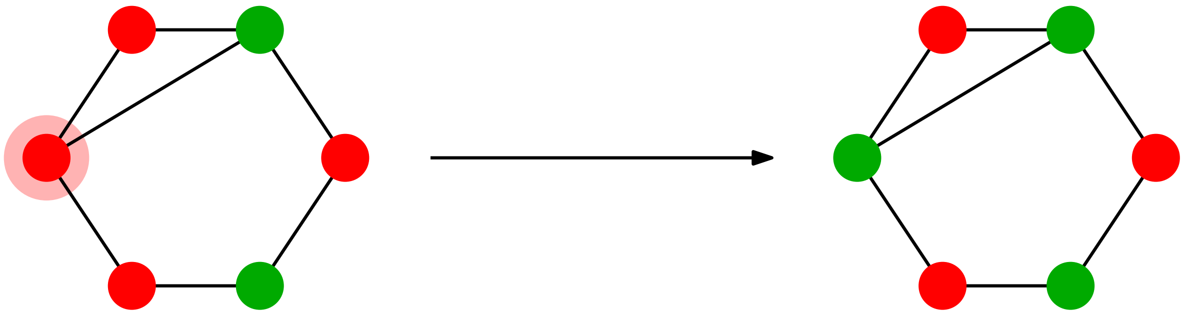


- **Convergence?**

Fixing procedure: *pick an arbitrary invalid node and flip its color*



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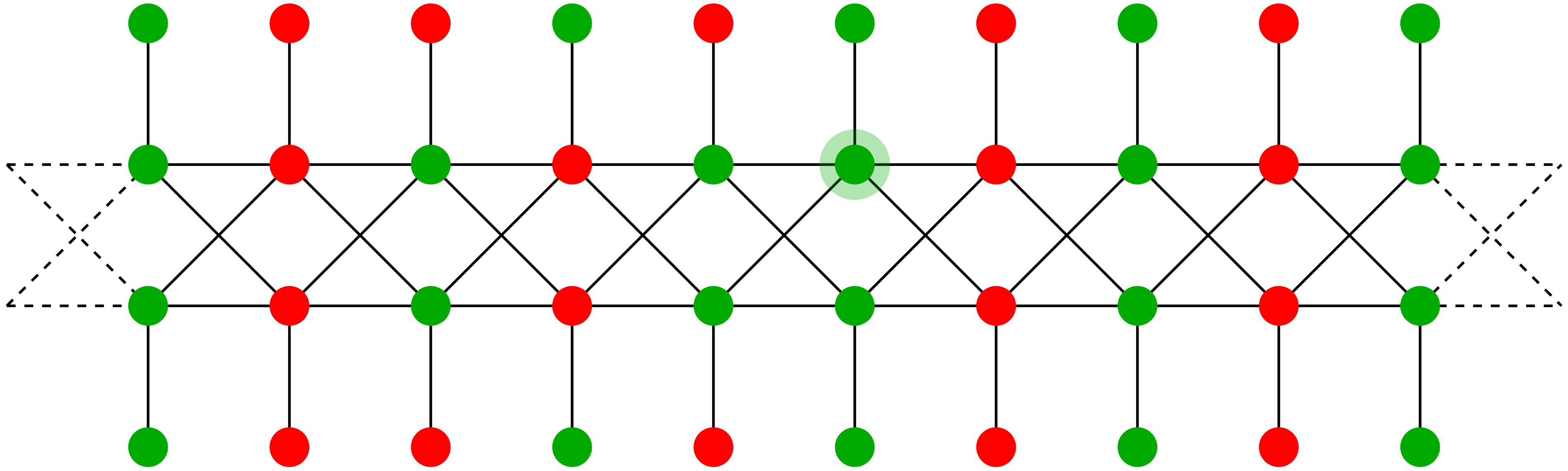

- **Convergence?** Potential function!

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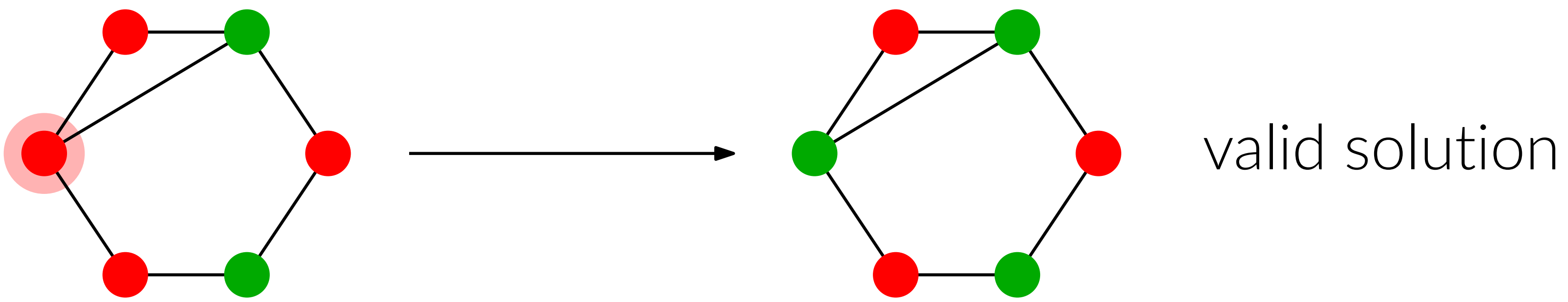

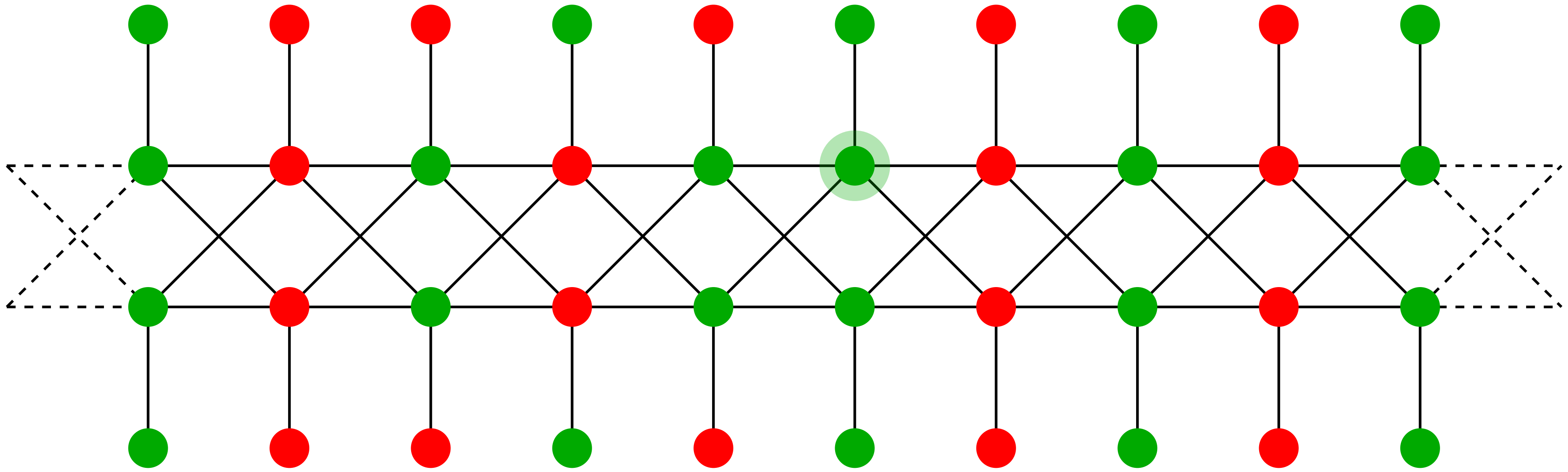

- **Convergence?** Potential function!

  - # of monochromatic edges decreases

LOC: fixing procedure

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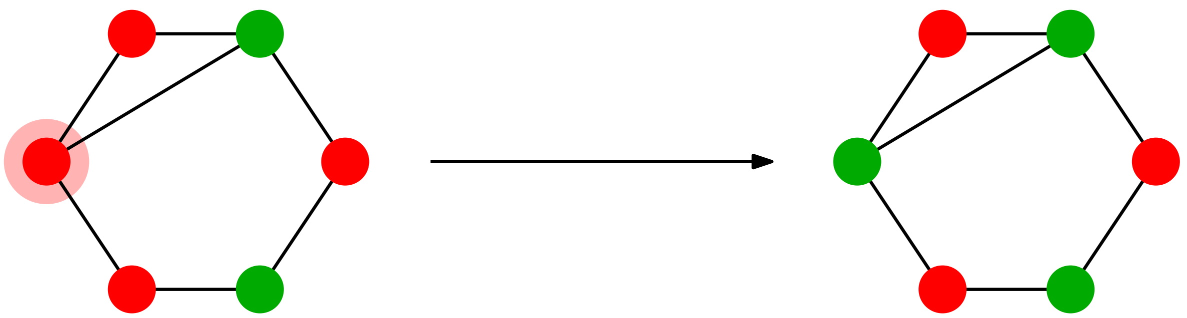

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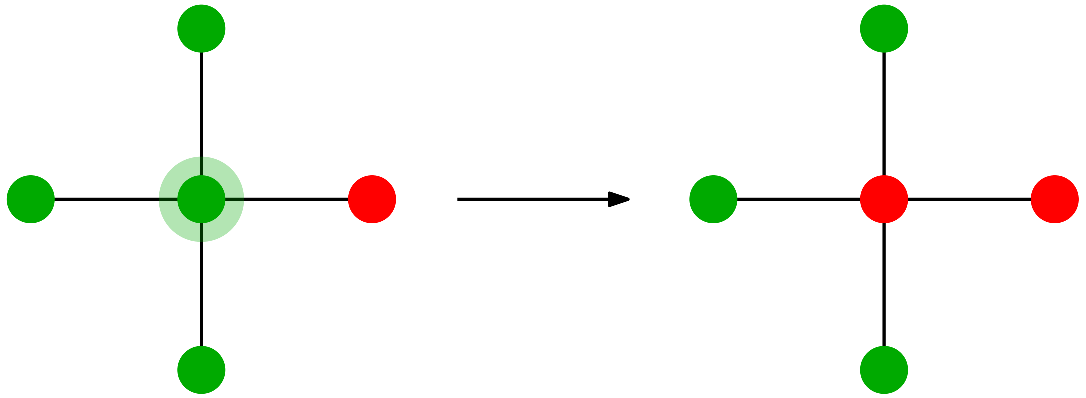

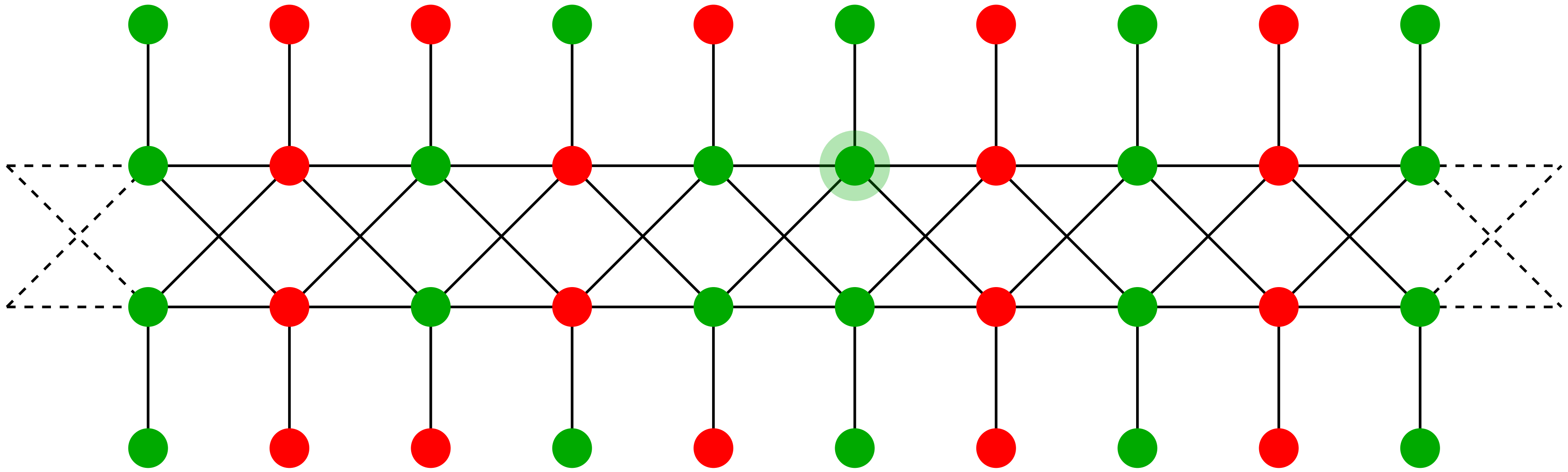

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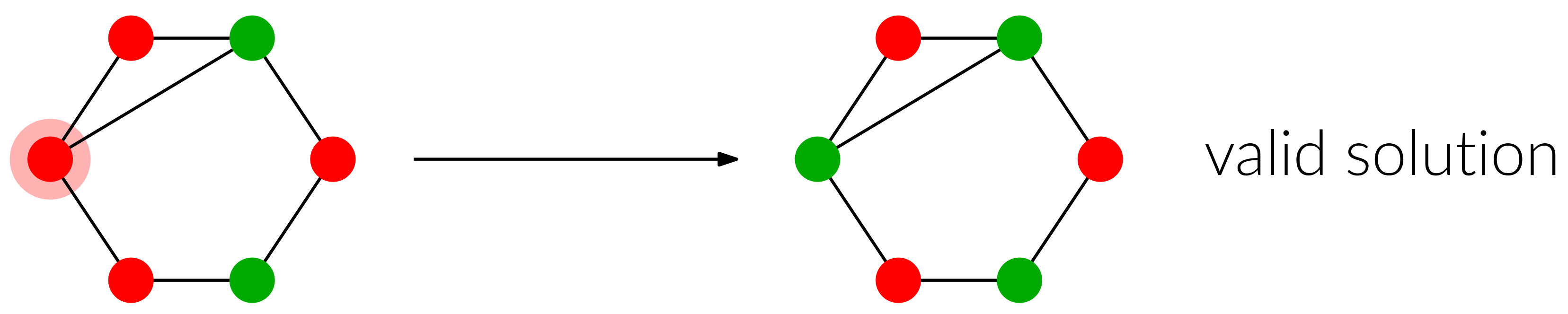

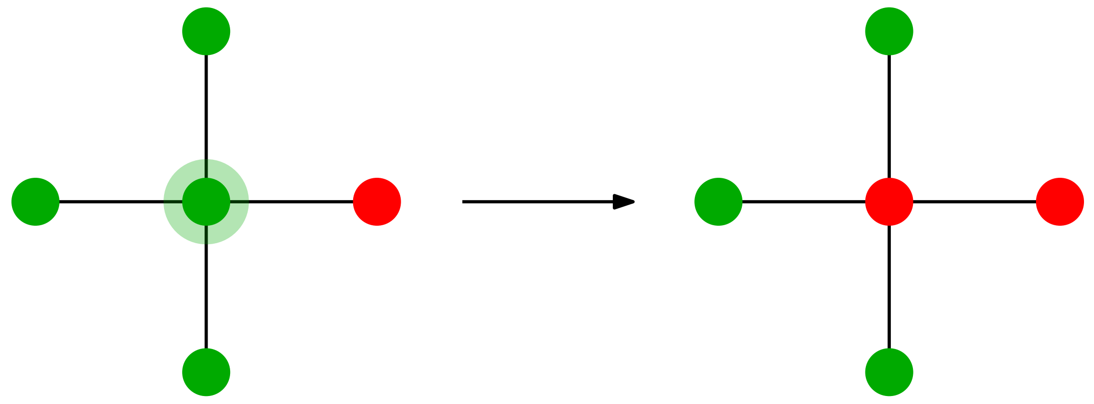

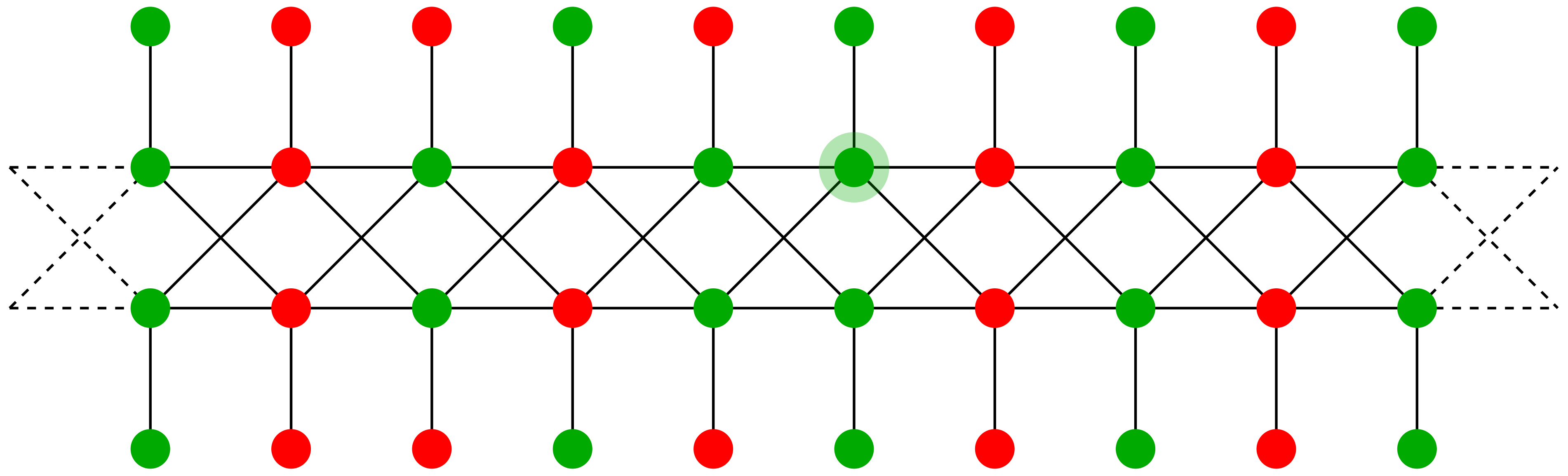

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- Possibly, very long chains…

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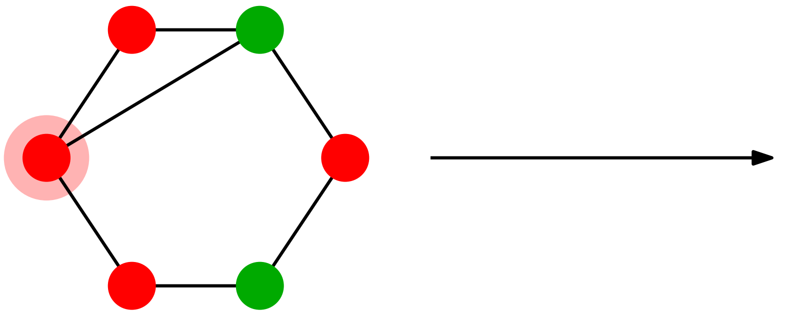

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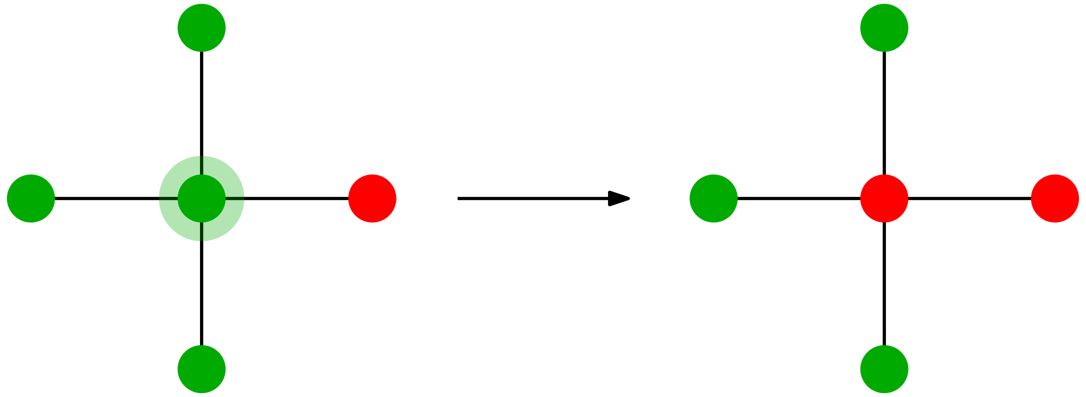
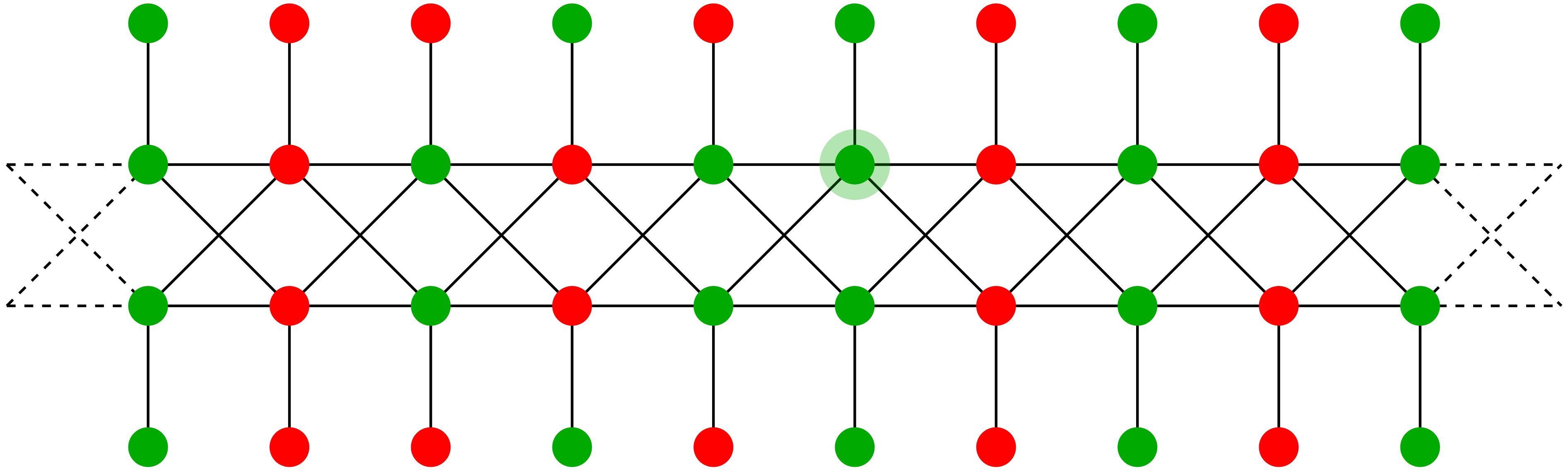




- **Convergence?** Potential function!

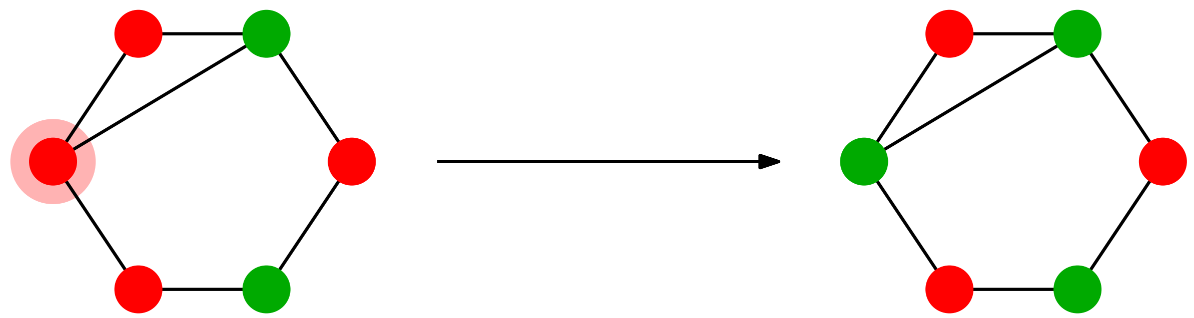
  - # of monochromatic edges decreases

- Possibly, very long chains…

  - $O(|E|)$-time sequential algorithm

LOC: fixing procedure

Fixing procedure: *pick an arbitrary invalid node and flip its color*



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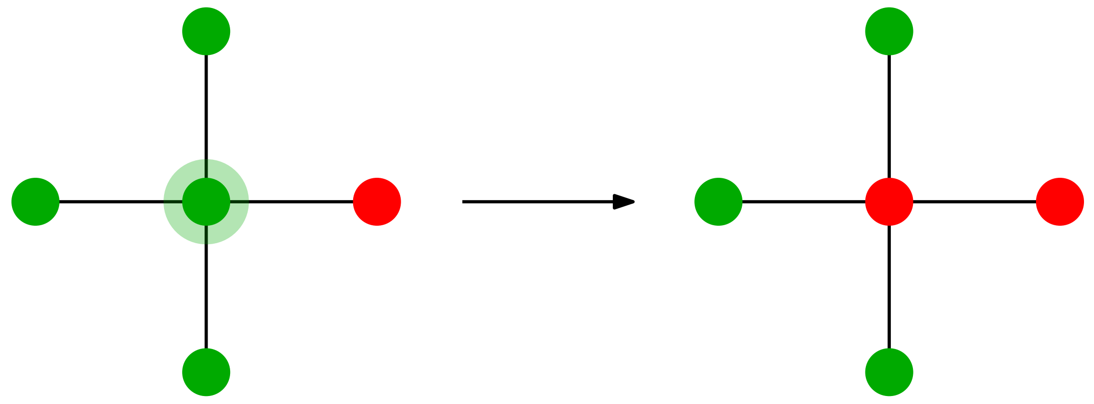

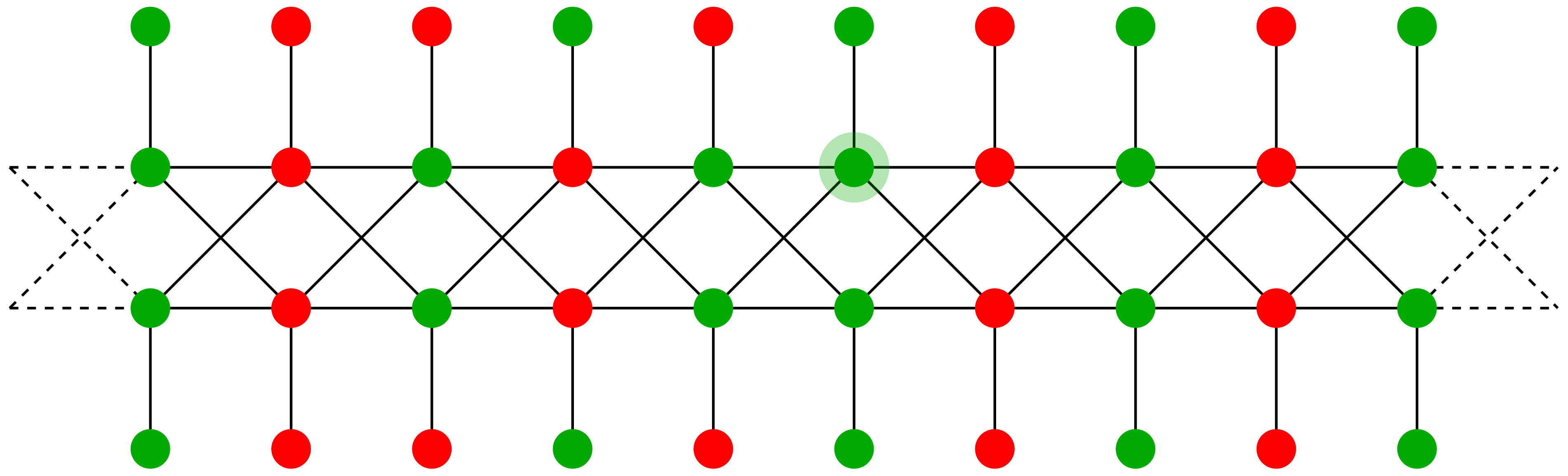

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- With distributed algorithms?

[Linial FOCS '87 & SICOMP '92]

- **Distributed network** of n processors/nodes
  - graph $G = (V, E)$ with $|V| = n$
  - E: communication links
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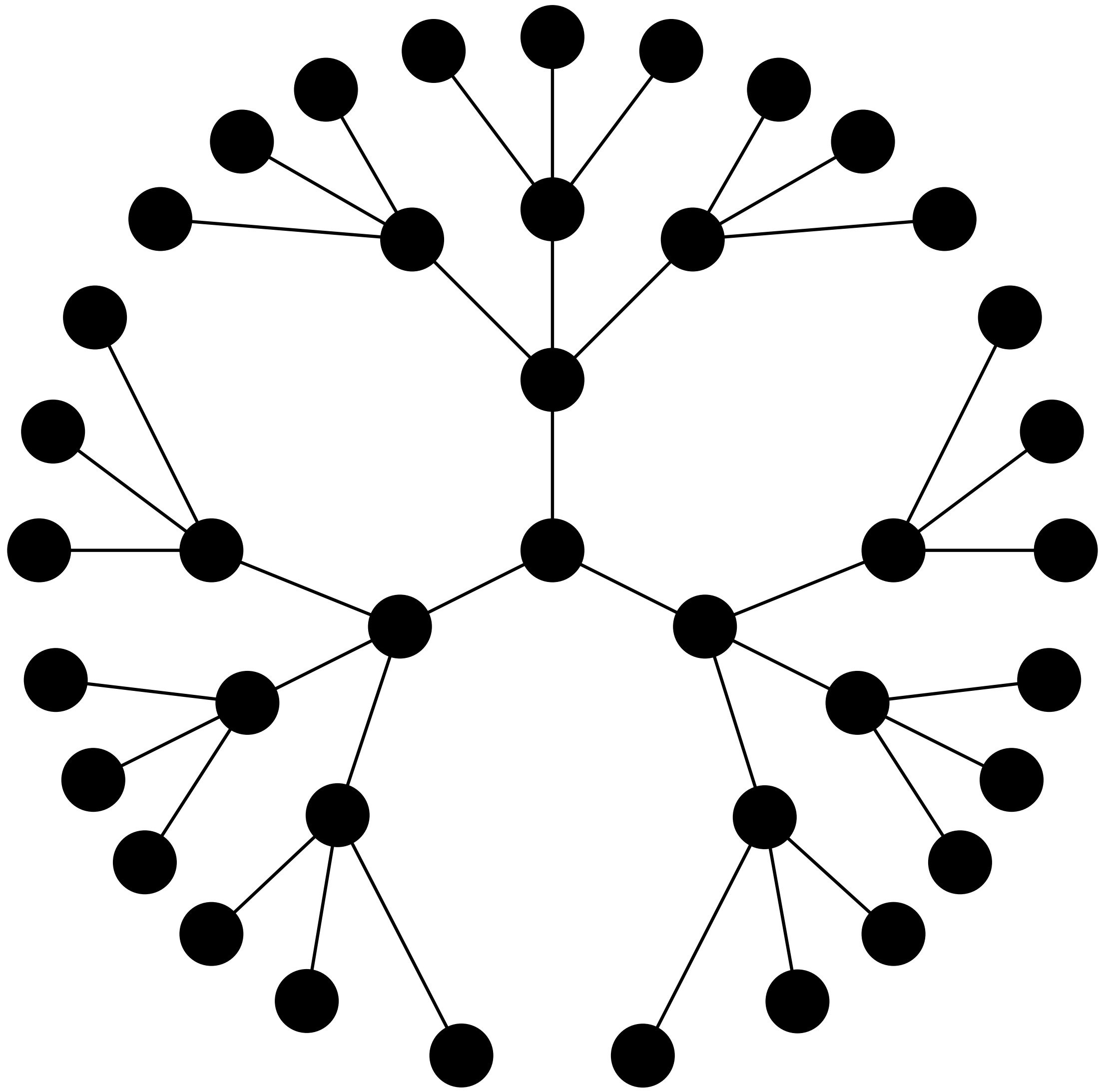

The LOCAL model

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  - arbitrary local computation & update of state variables
  - sending of messages to all neighbors
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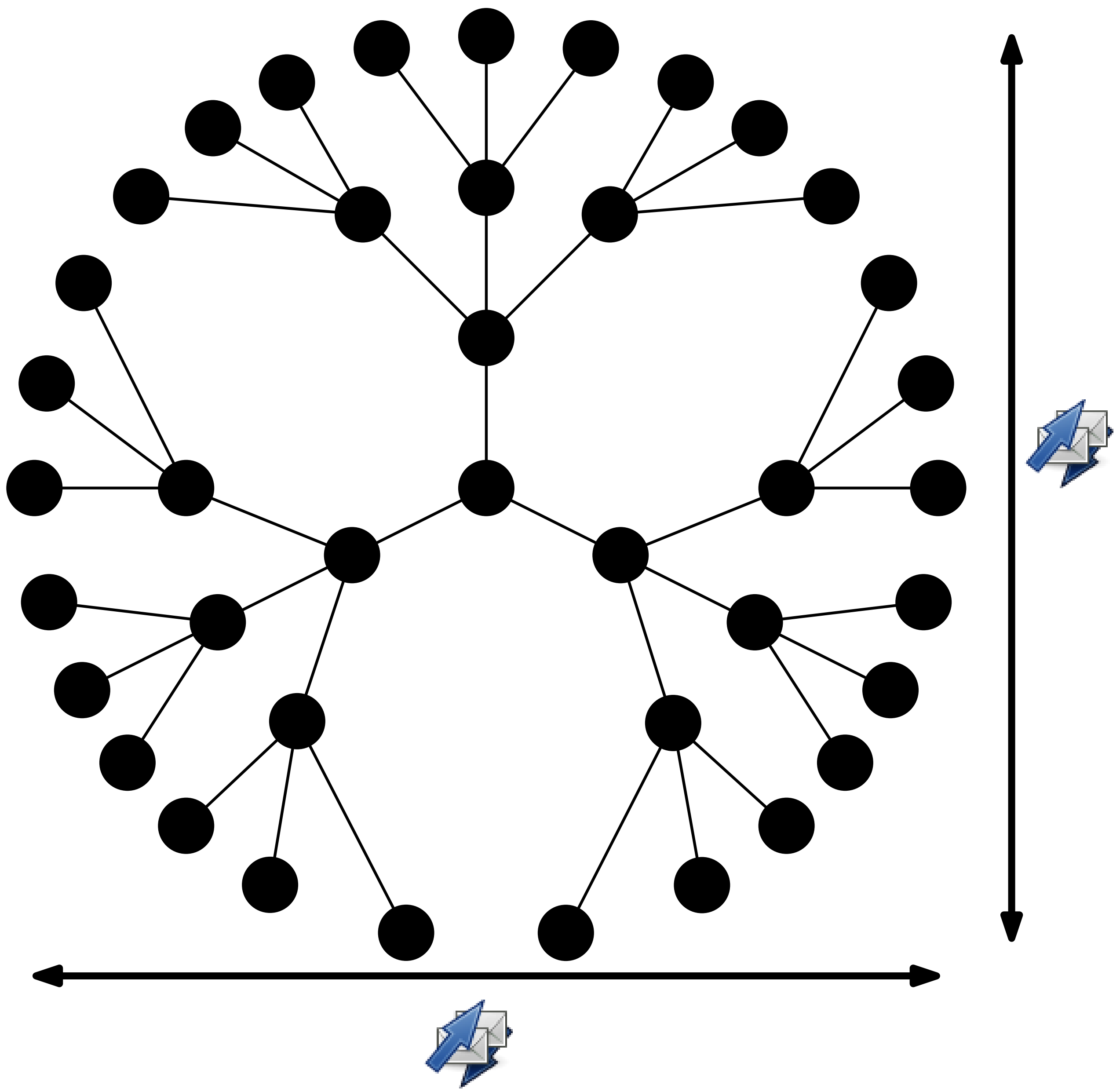

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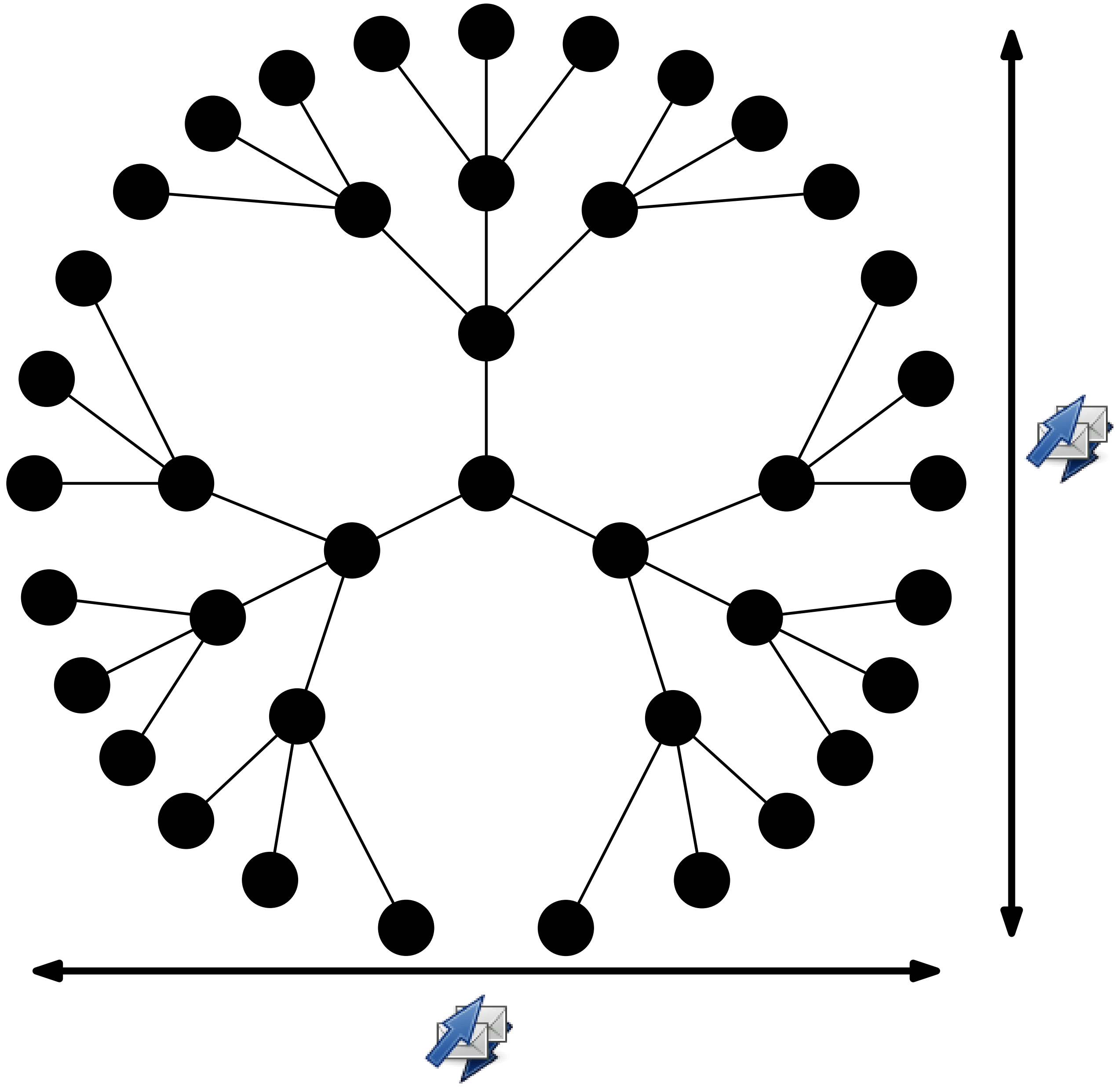
  - needed to solve even basic problems (2-coloring a 2-path)



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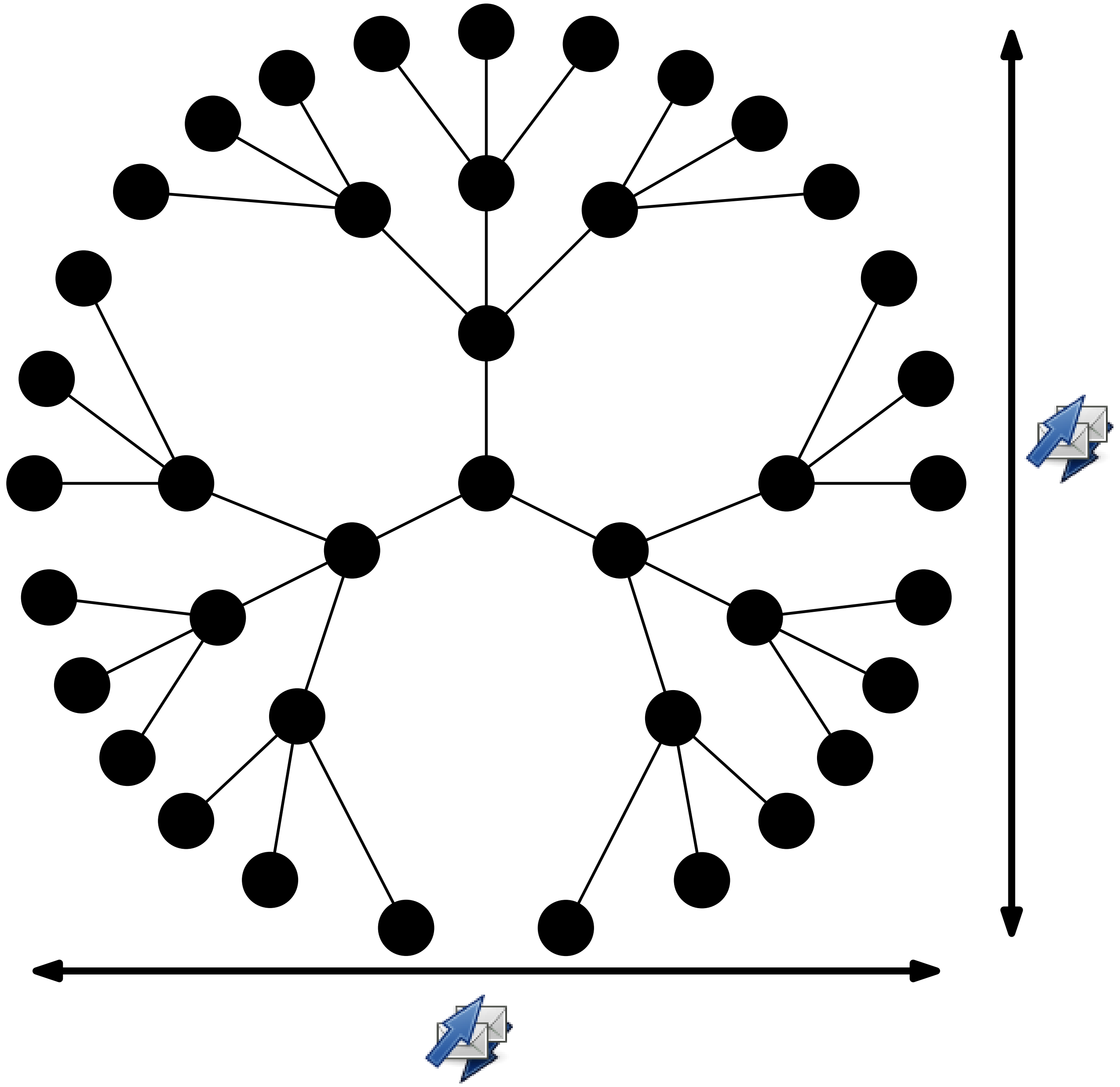

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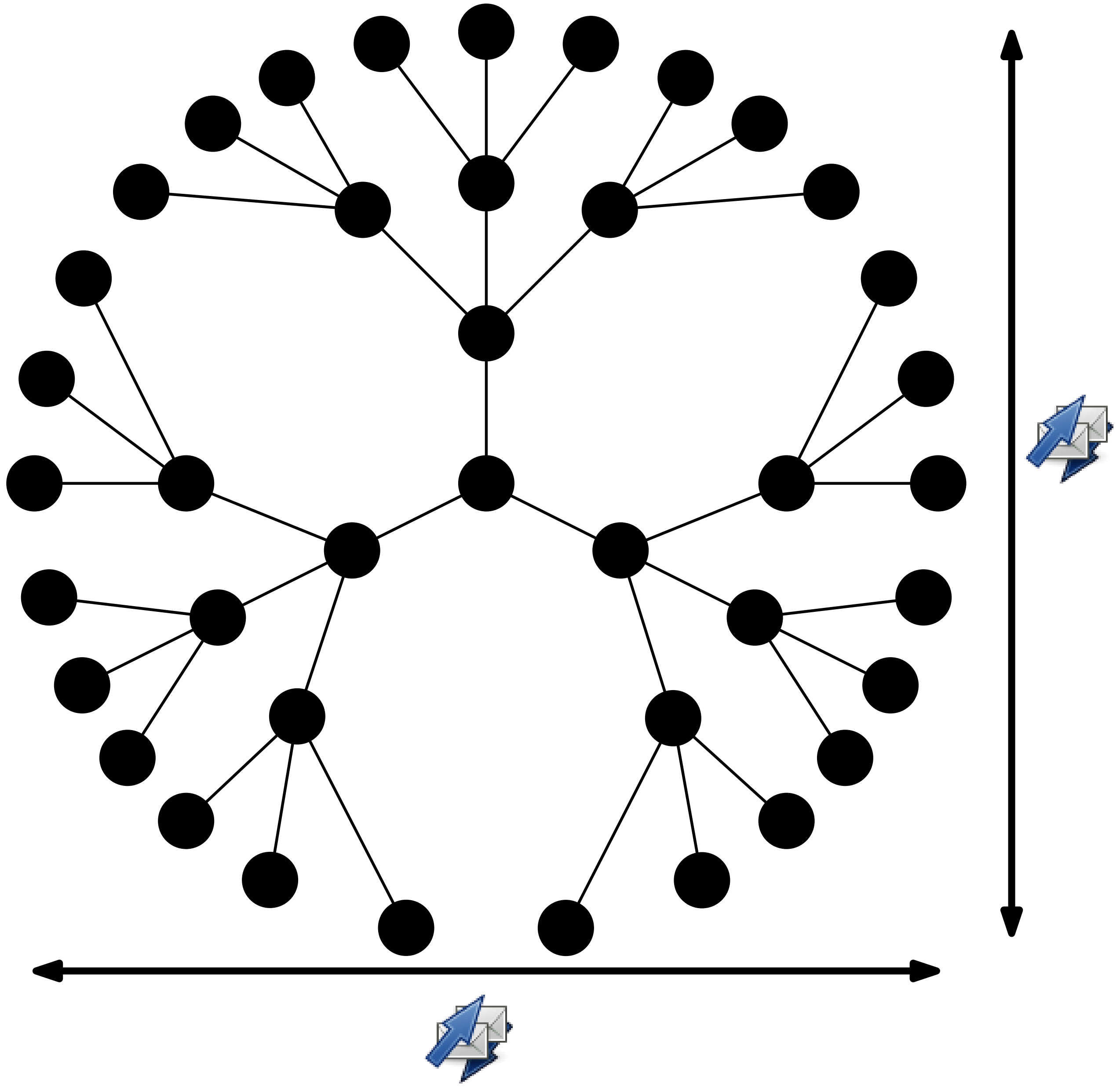
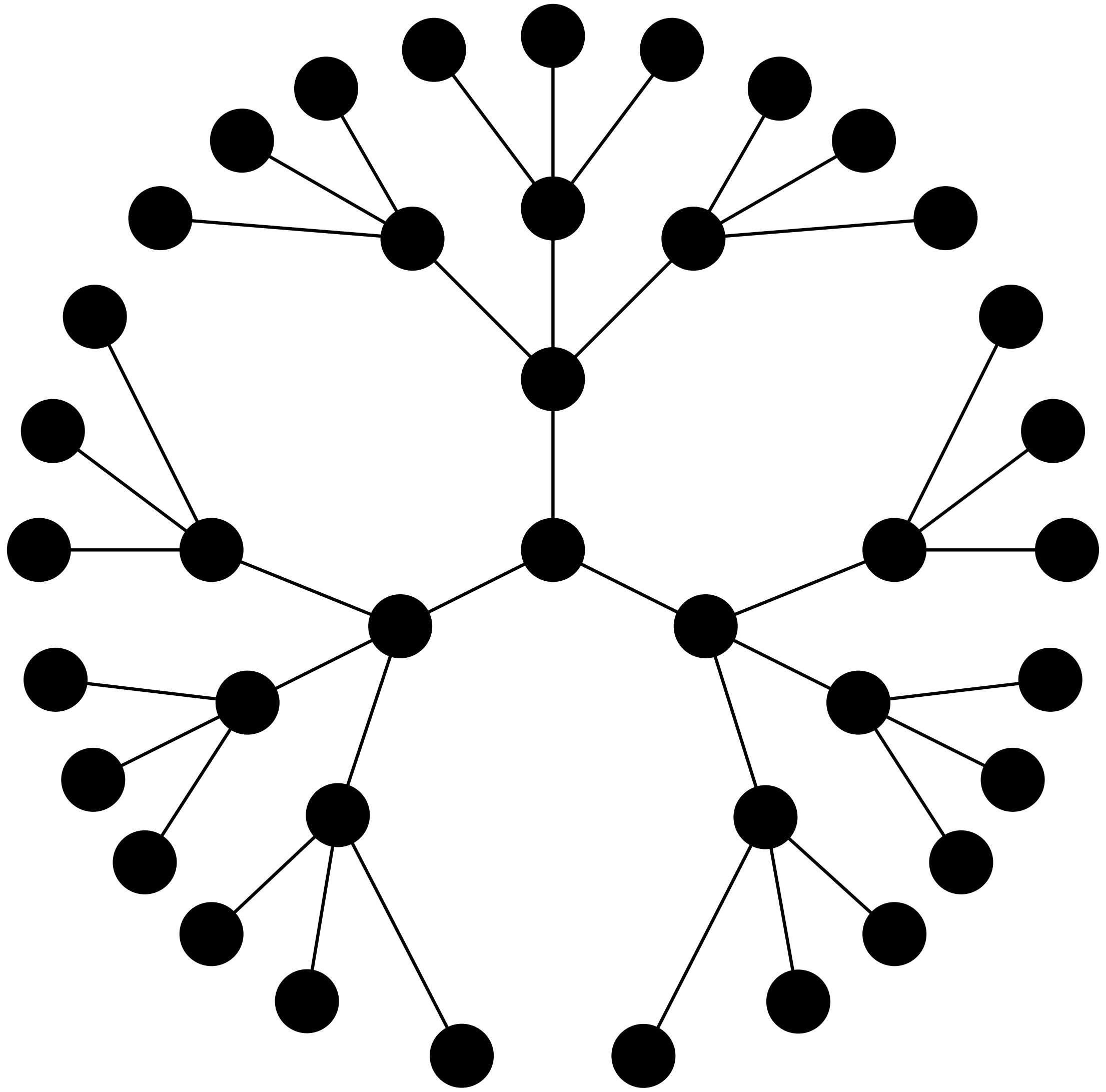


Local view

Complexity measure: number of communication rounds



- What do we know after T rounds?

Local view

Complexity measure: number of communication rounds

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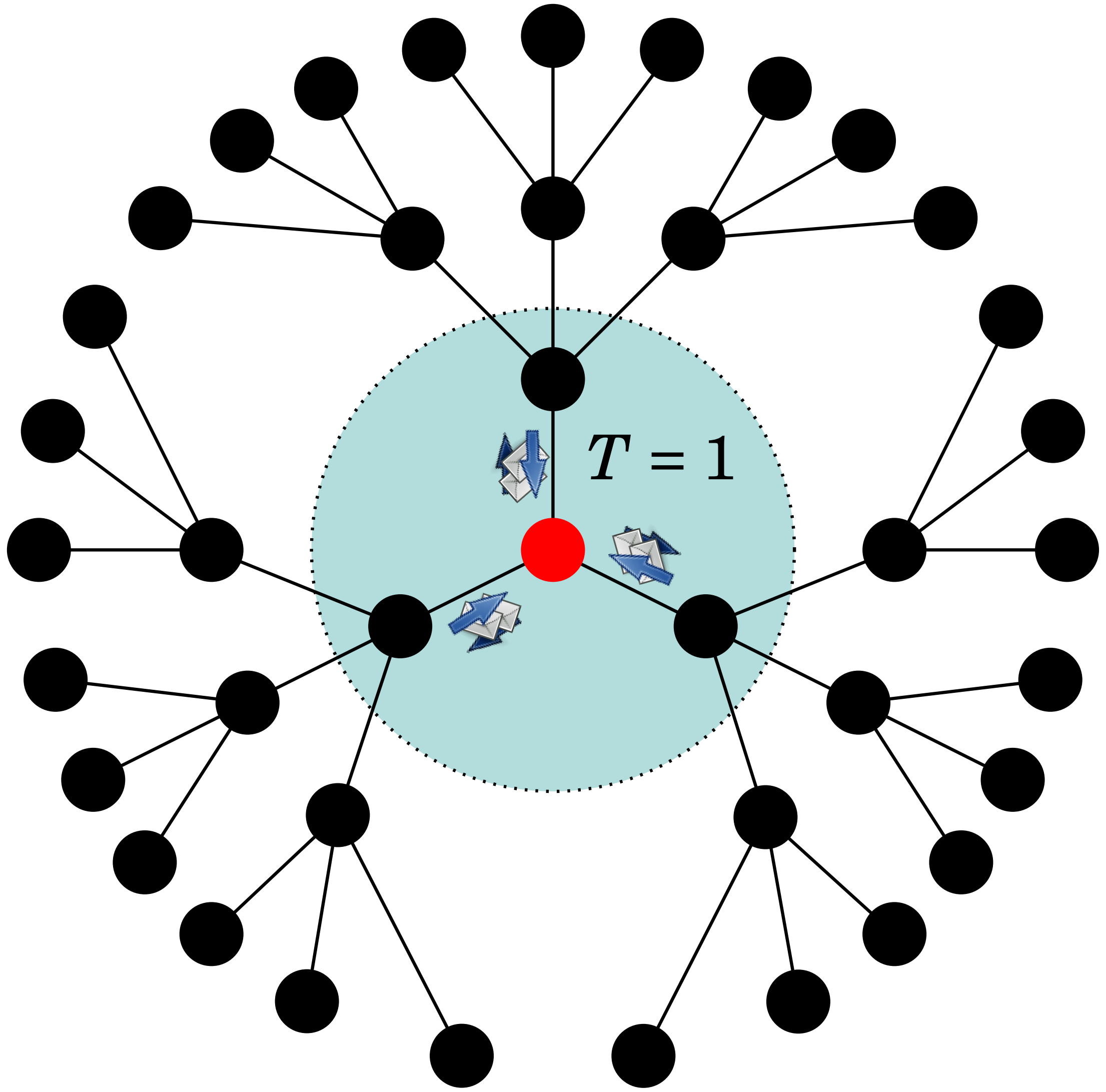

knowledge after $\boldsymbol{T}$ rounds of communication

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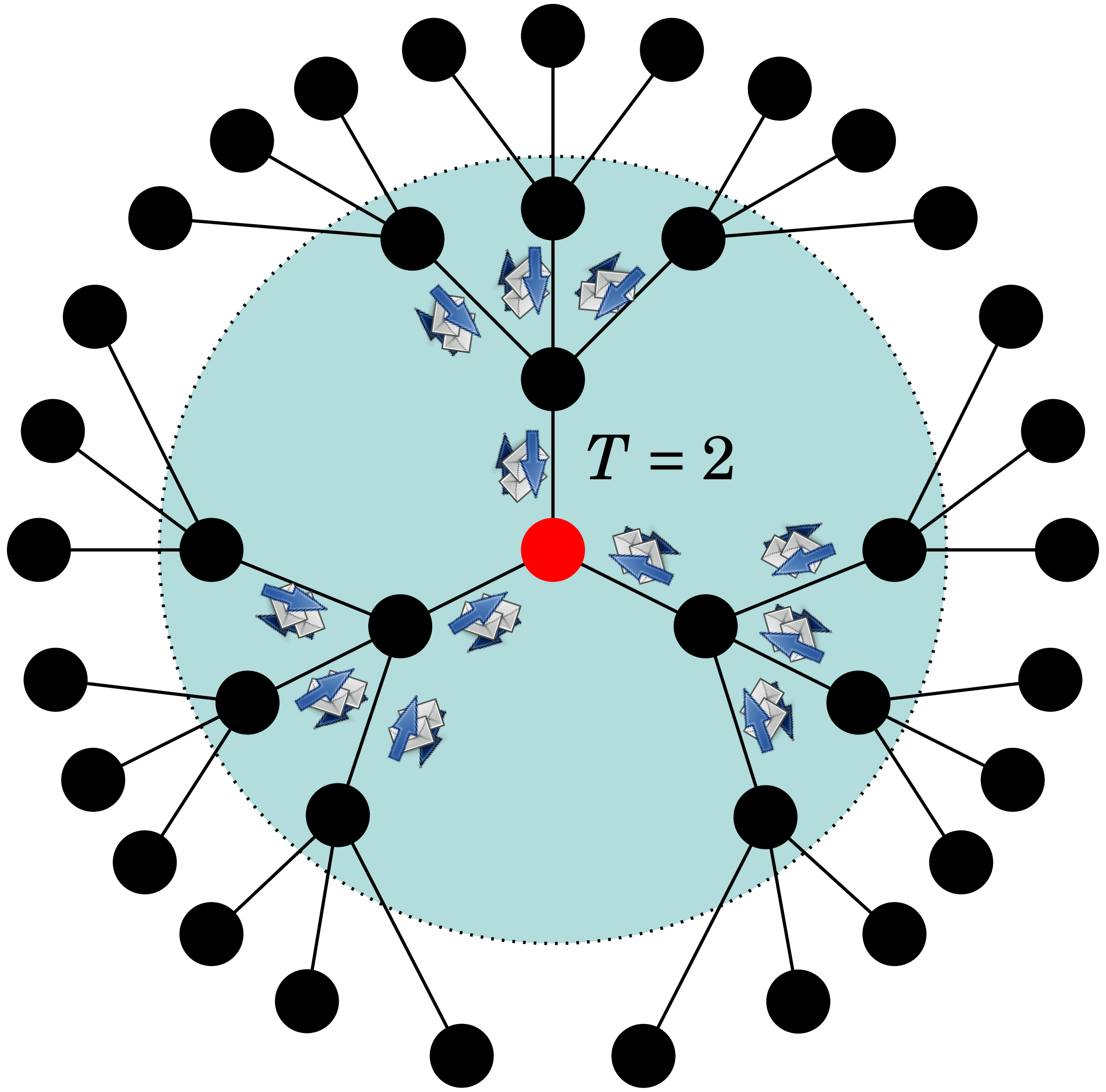

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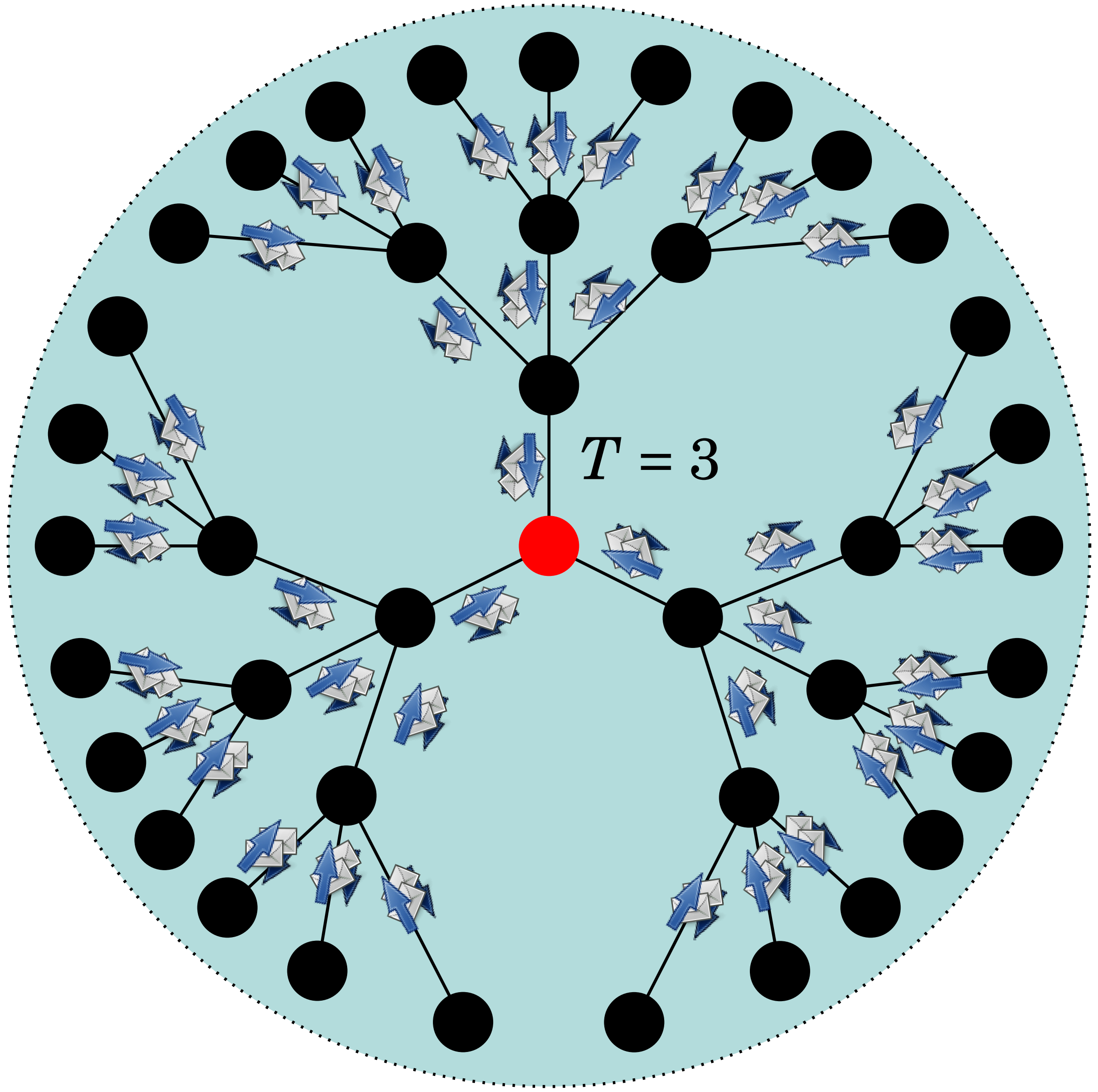

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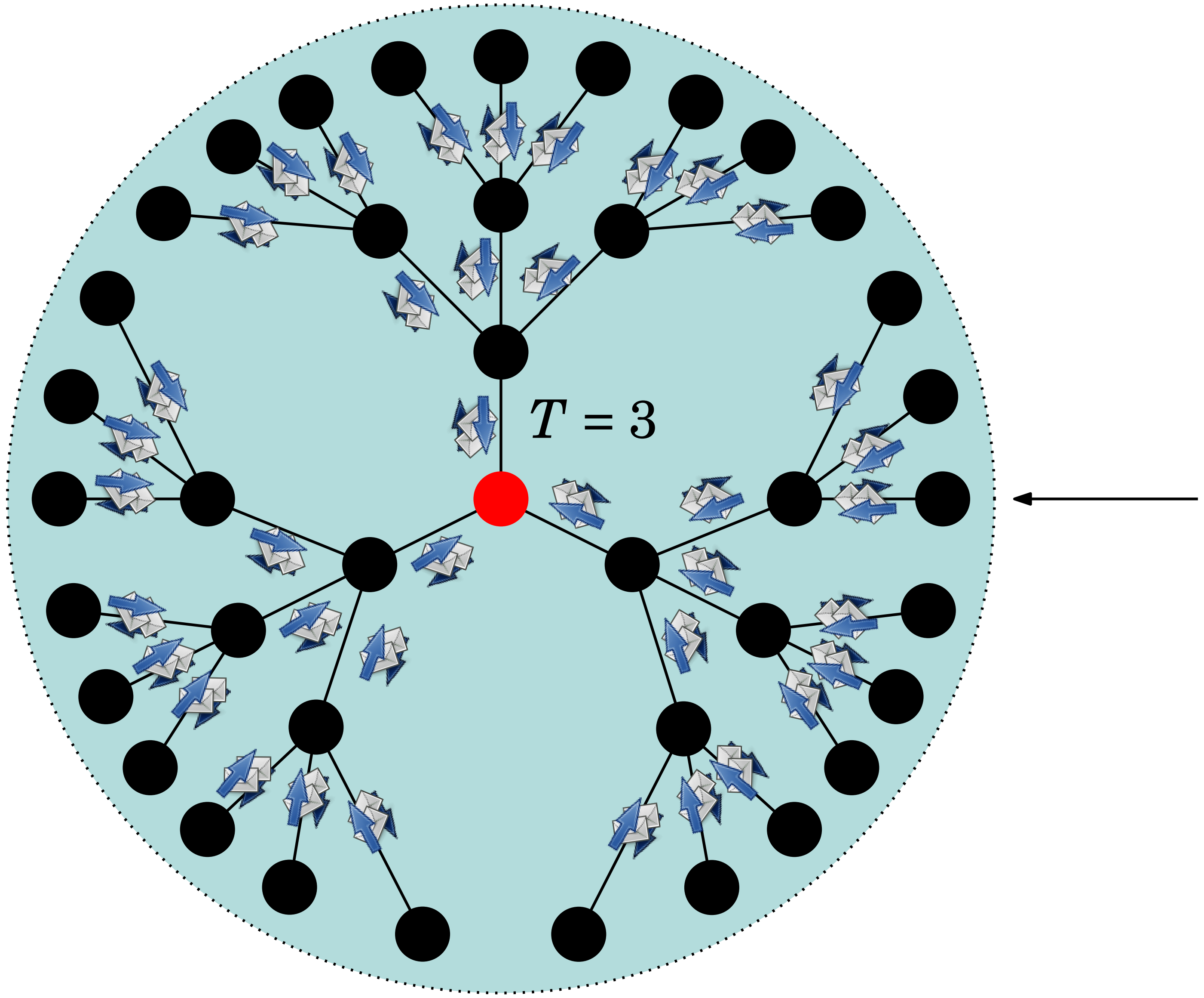

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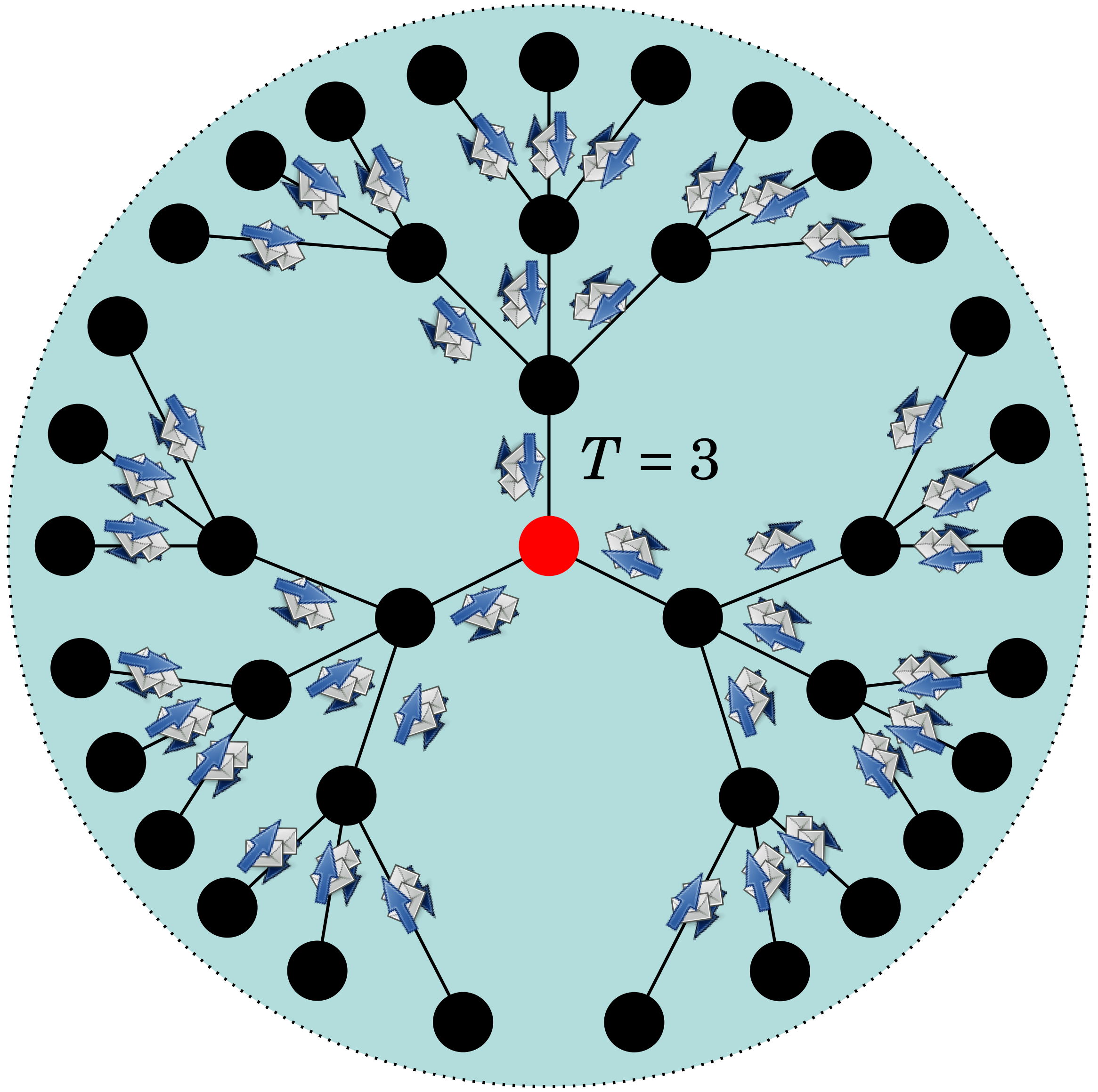

knowledge after T rounds of communication

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- LOCAL algorithm A with locality T_A + LOCAL algorithm B with locality T_B = LOCAL algorithm C with locality $T_A + T_B$

Previous results about LOC

Lower bound: - $\Omega(\log n)$-rounds in deterministic LOCAL (in *bounded-degree* trees)

- $\Omega(\log \log n)$-rounds in randomized LOCAL (in *bounded-degree* trees)

- reduction from Sinkless Orientation [Balliu, Hirvonen, Lenzen, Olivetti, and Suomela, SIROCCO '19]
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HUGE GAP!

Let's find a *better* distributed algorithm...
(for bounded-degree graphs)

MPX subroutine

(α, d)-**decomposition** of a graph $G = (V, E)$:

- partition of V into clusters (sets) $C_1, \ldots, C_k$
- $\mathrm{diam}(C_i) \leq d$ for all i
- # inter-clusters edges $\leq \alpha |E|$

G

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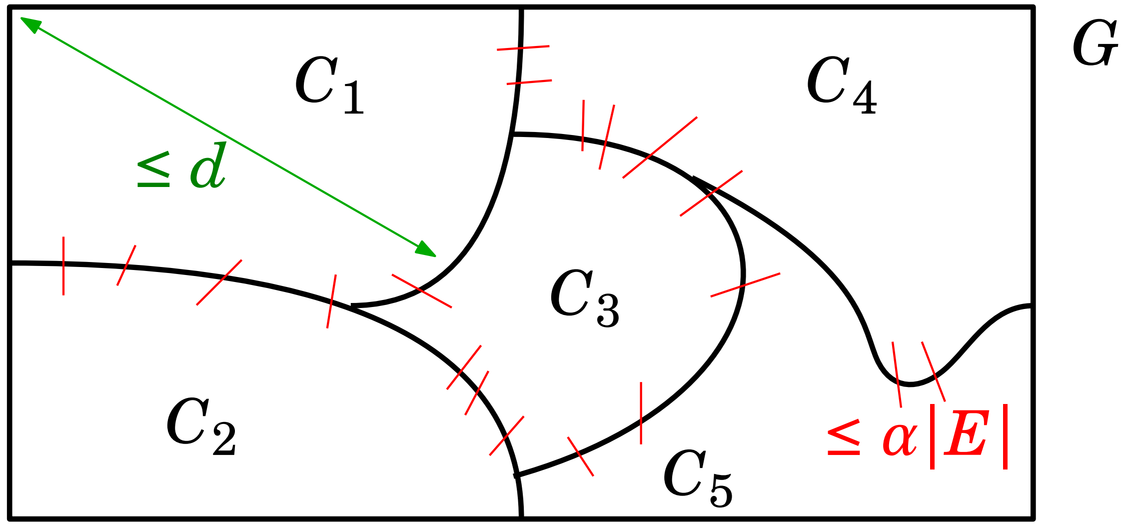

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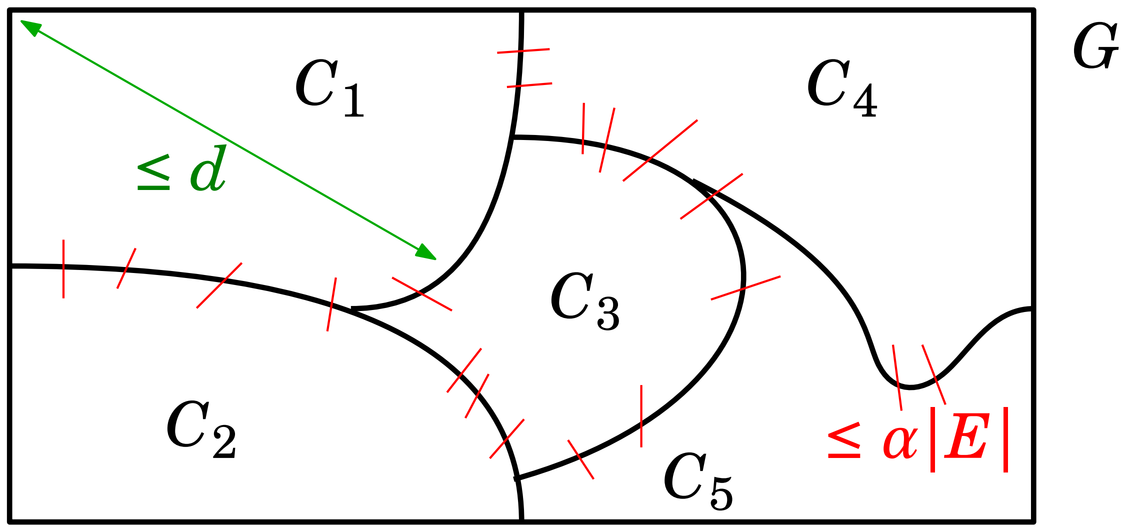
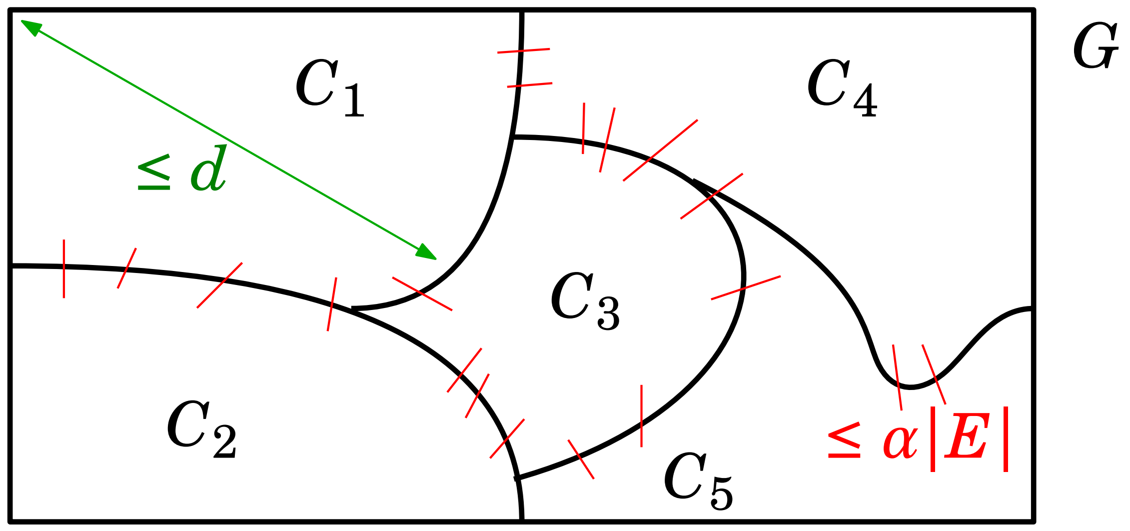


Theorem (adaptation of [Miller, Peng, and Xu, SPAA '13]):

There exists a randomized LOCAL algorithm $\mathcal{MPX}$ that computes an (α, d)-decomposition of a graph $G = (V, E)$ with the following properties:

- *Running time $O(\log n/\alpha)$.*
- *UB on the diameter is $d = O(\log n/\alpha)$.*
- *For each $v \in V$, with probability $\geq 1/2$ it holds that $\mathcal{N}_{\Theta(1/\alpha)}[v] \subseteq C_i$ for some i.*

A "simple" algorithm



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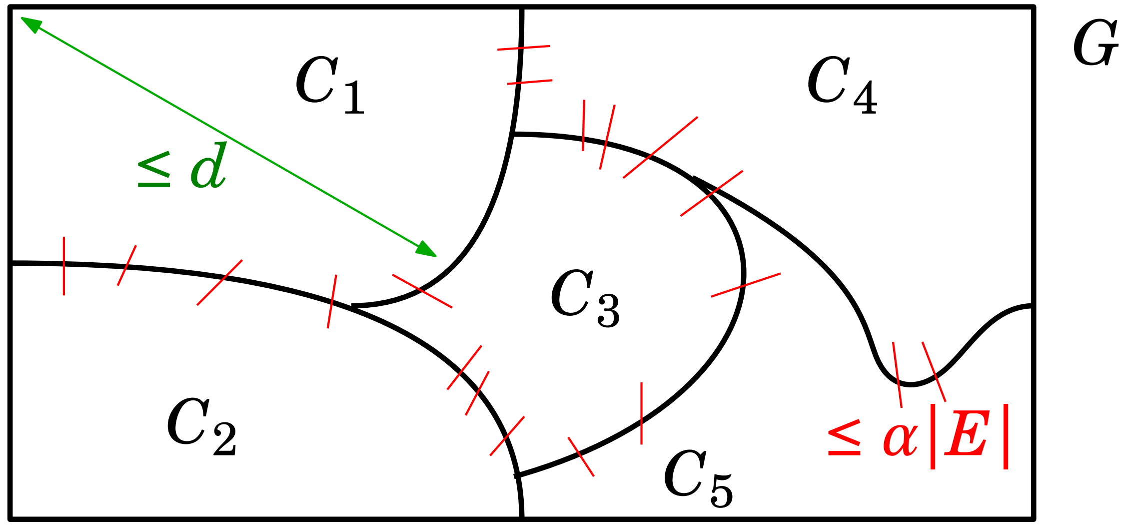

- *Brute-force solution with minimum potential in each cluster (time* $\boldsymbol{O(d)}$*)*

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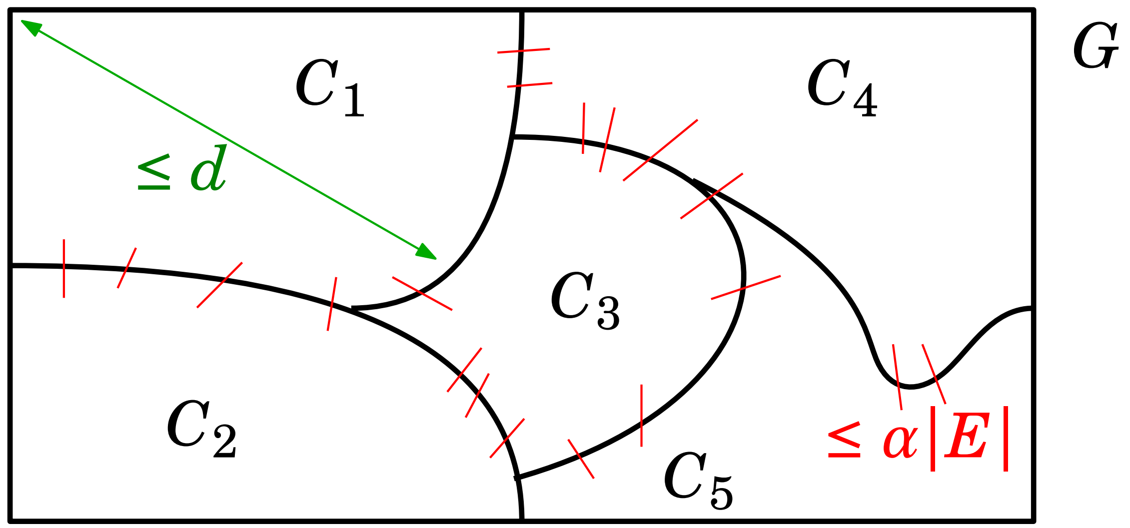

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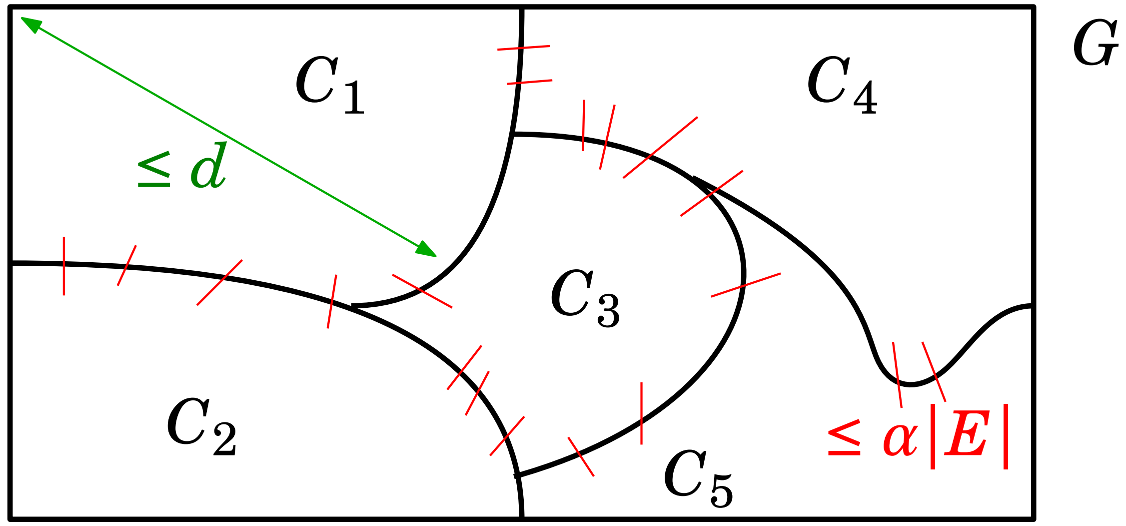

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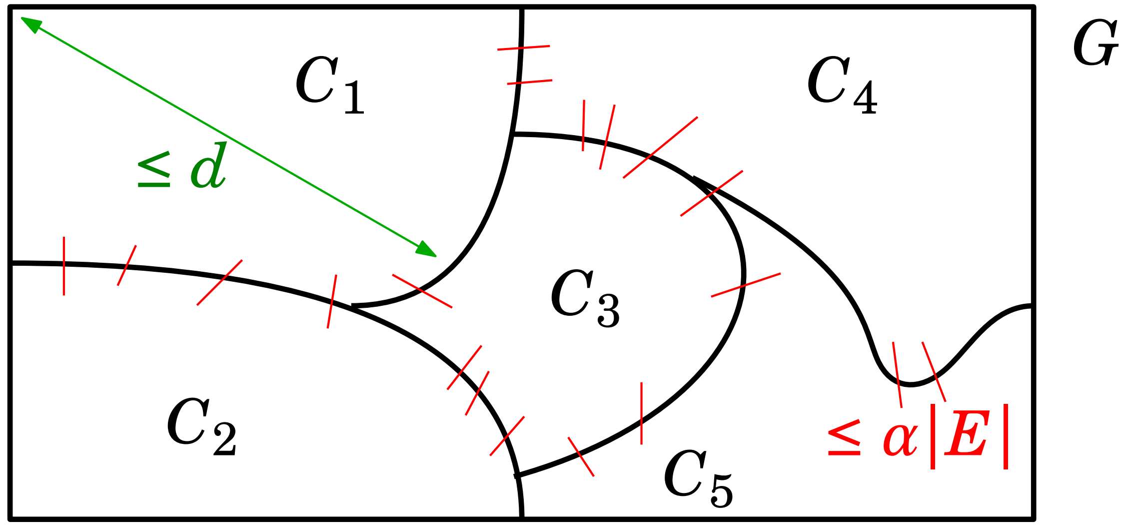

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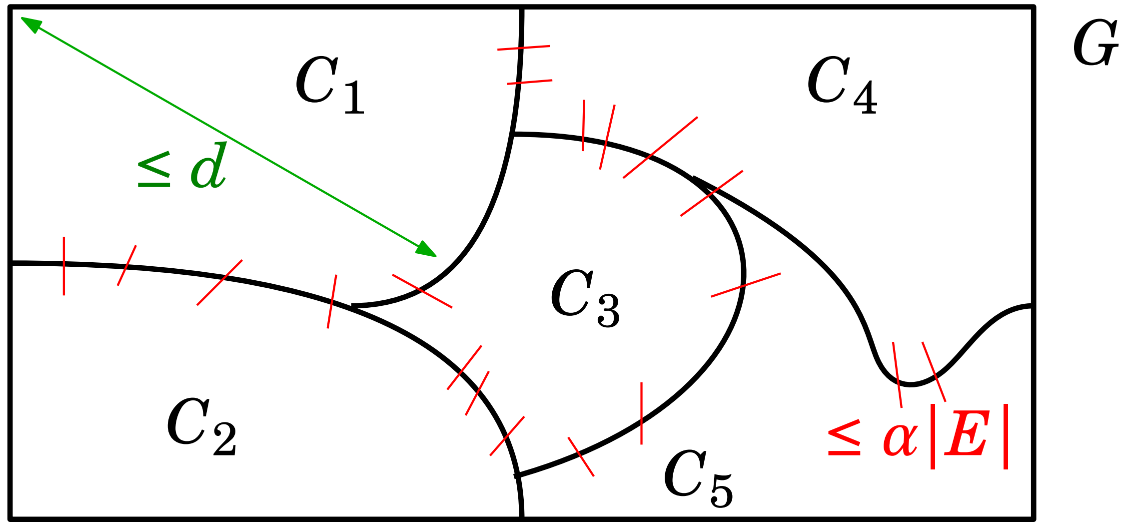

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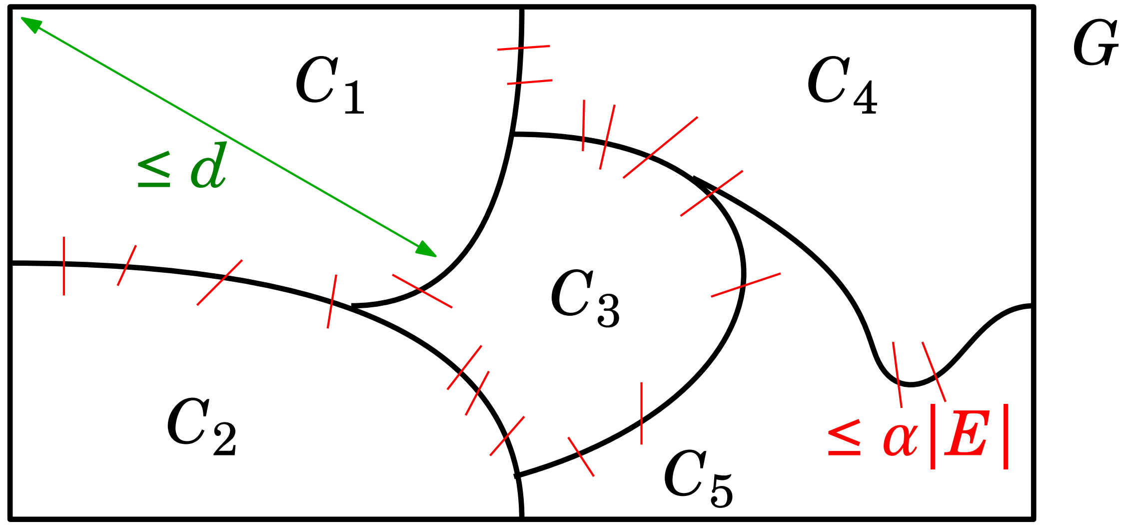

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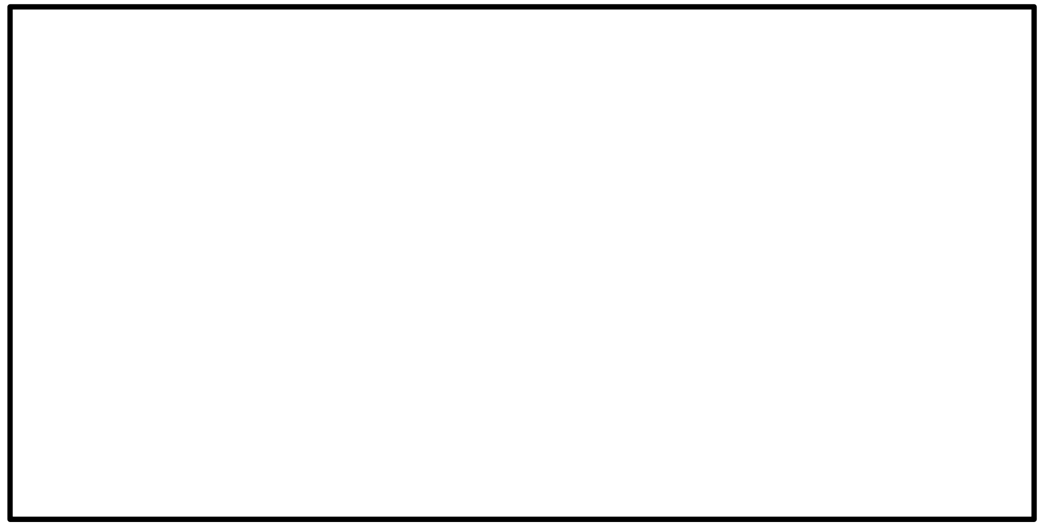
- bounded-degree graphs: running time $O(\sqrt{n \log n})$ (minimized by $\alpha = \sqrt{\log n/n}$)
- still far from the lower bounds … How to do better?

Repeating “brute-force” does not work



G

- *Repeat:*

  - *Run $\mathcal{MPX}$ to get (α, d)-network decomposition (time $O(d)$)*

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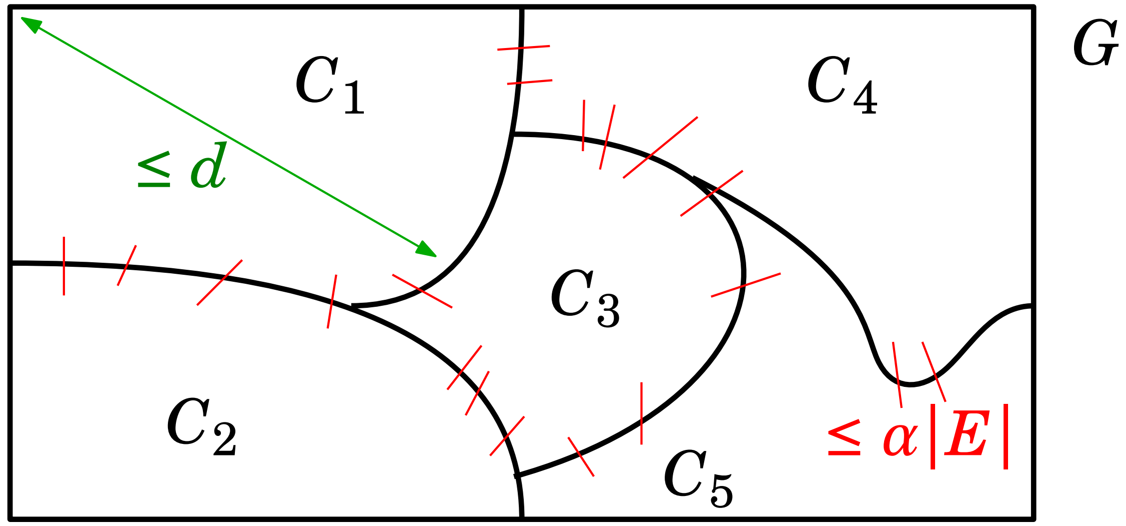

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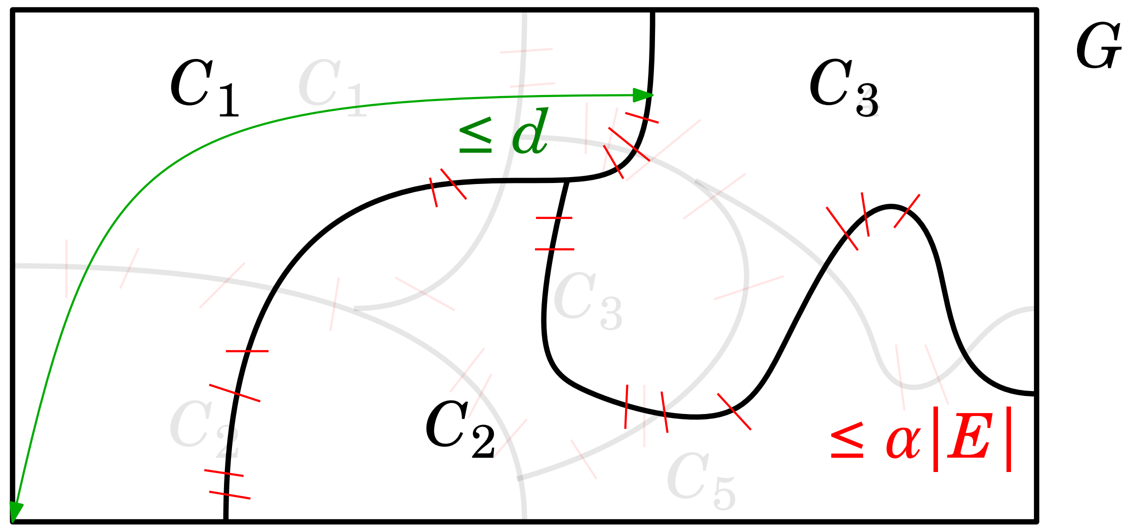

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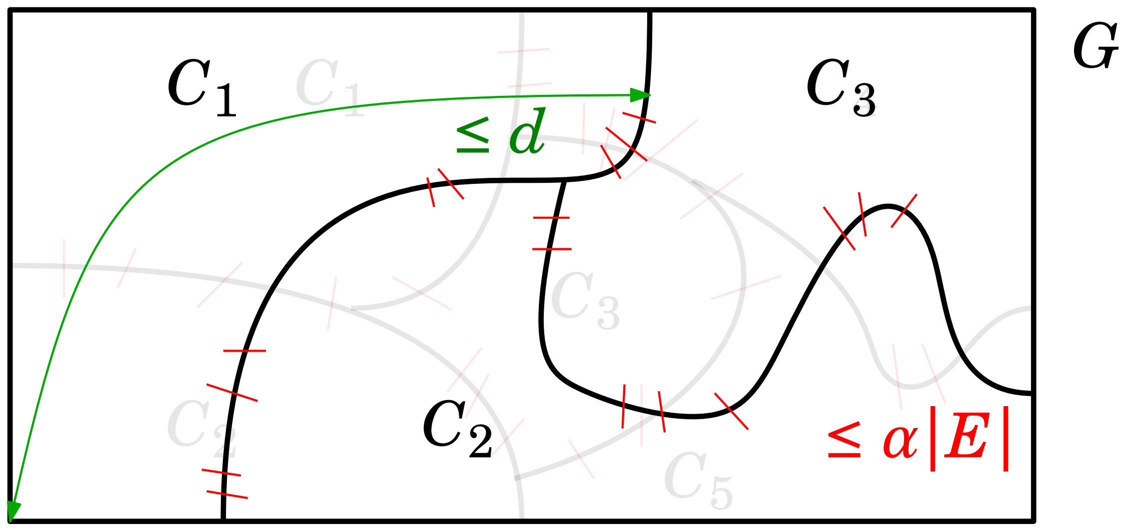

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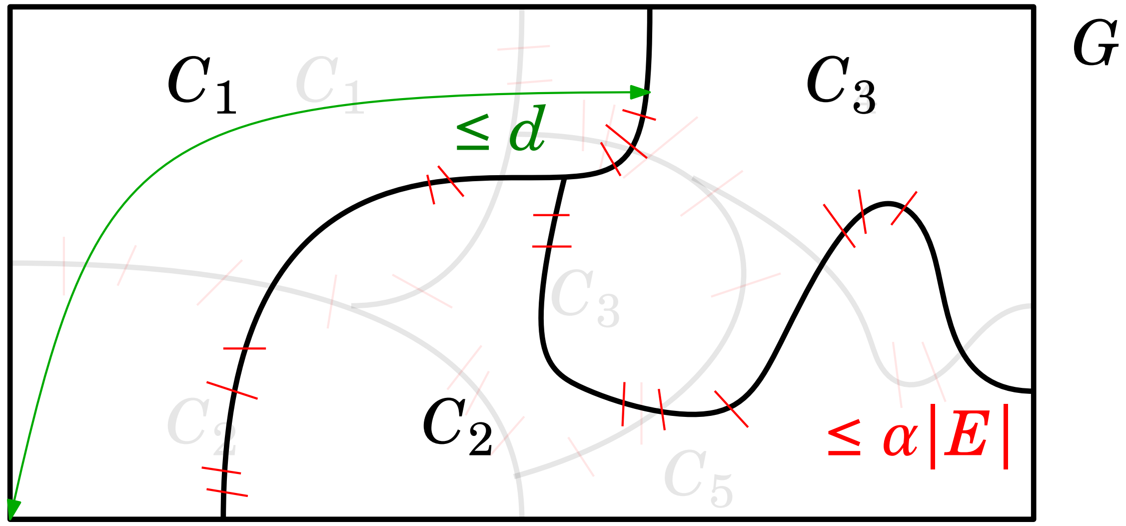

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  - what to do?

Improving sets

Improving set in a 2-colored graph $G = (V, E)$

- Subset $A \subseteq V$ such that by flipping the colors of nodes in A the potential decreases

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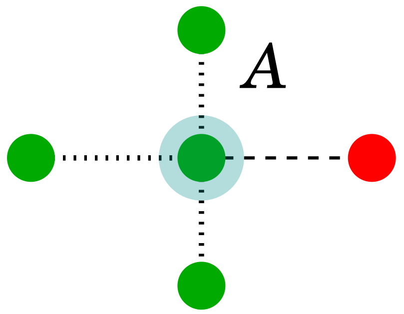

$\text{Imp}(A) = 3 - 1 = 2$

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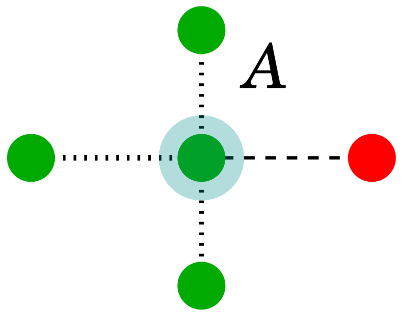

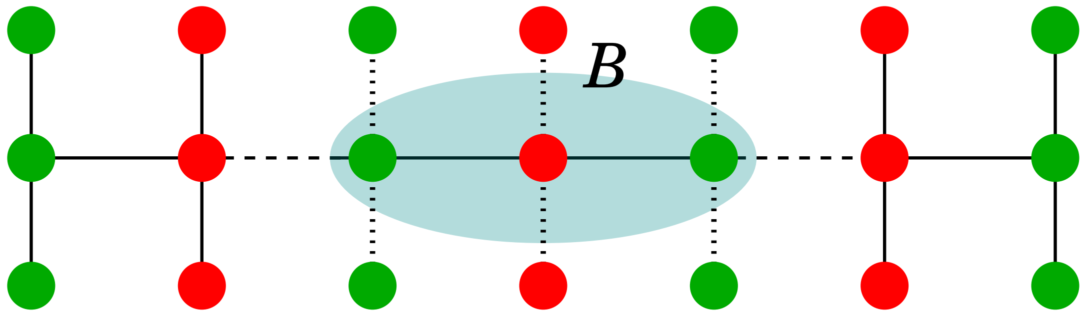

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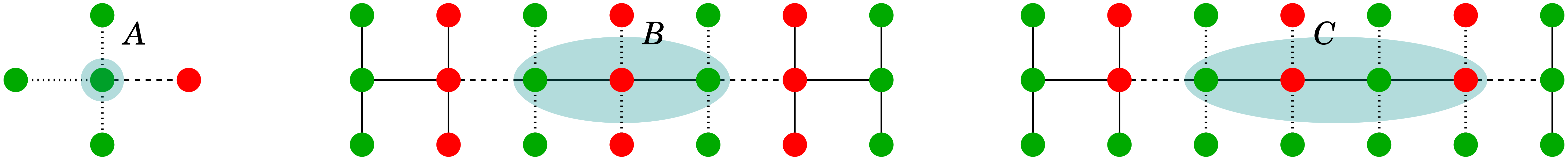

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$\text{Imp}(C) = 2$

$\text{IR}(C) = 1/2$

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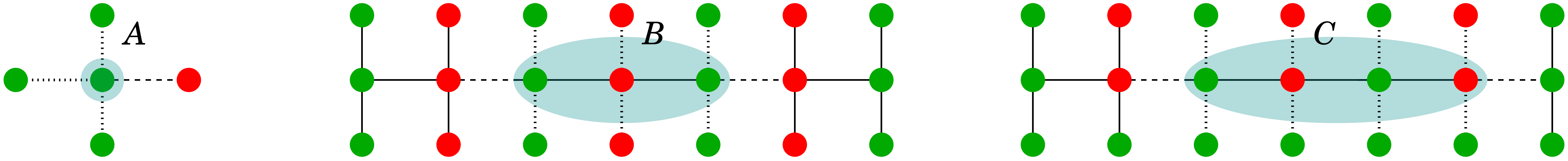

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Minimal improving set: improving set A such that

- There is no subset $A' \subseteq A$ with $\text{IR}(A') > \text{IR}(A)$ $\longrightarrow$ "quality": is there "useless stuff"?

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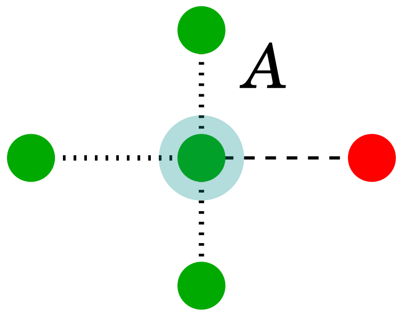
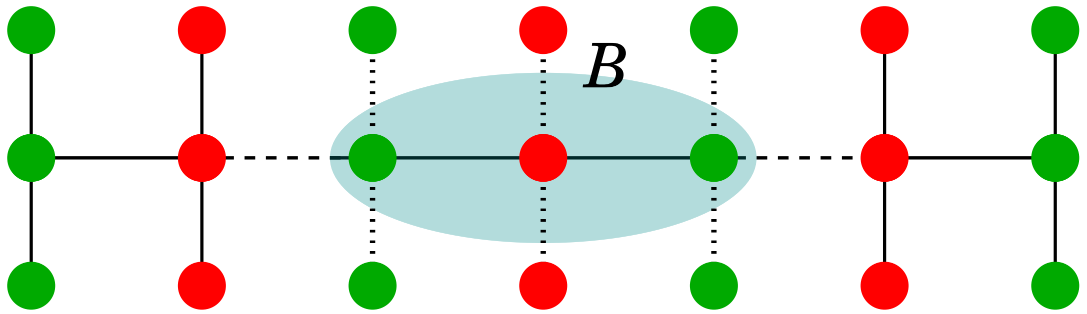
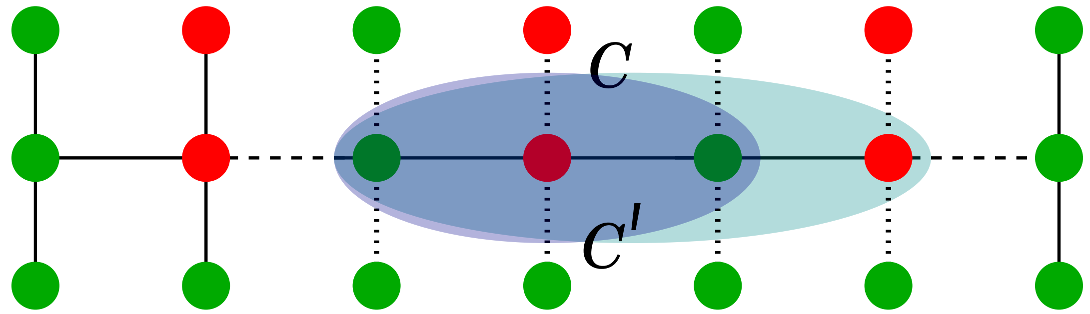
C not minimal







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$\text{IR}(C') = 2/3$

Minimal improving set: improving set A such that

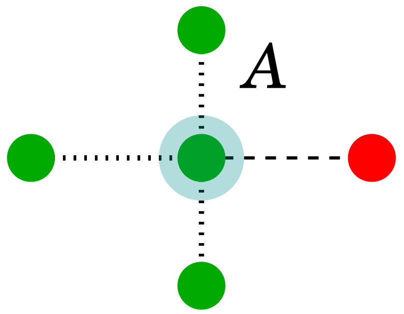
- There is no subset $A' \subseteq A$ with $\text{IR}(A') > \text{IR}(A)$ $\longrightarrow$ "quality": is there "useless stuff"?

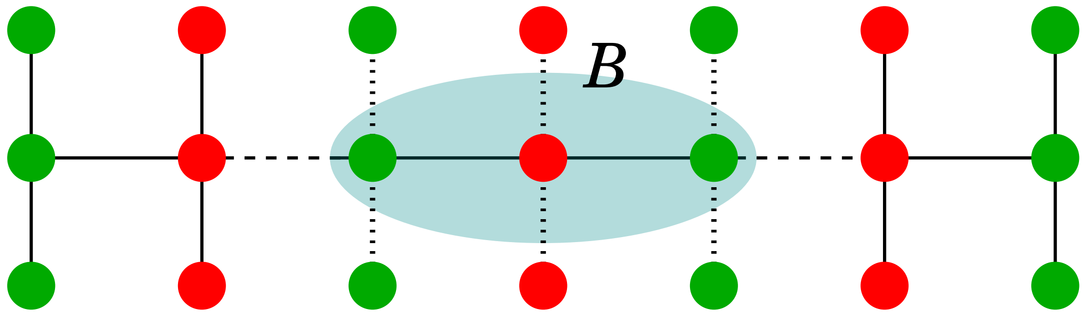
Improving sets

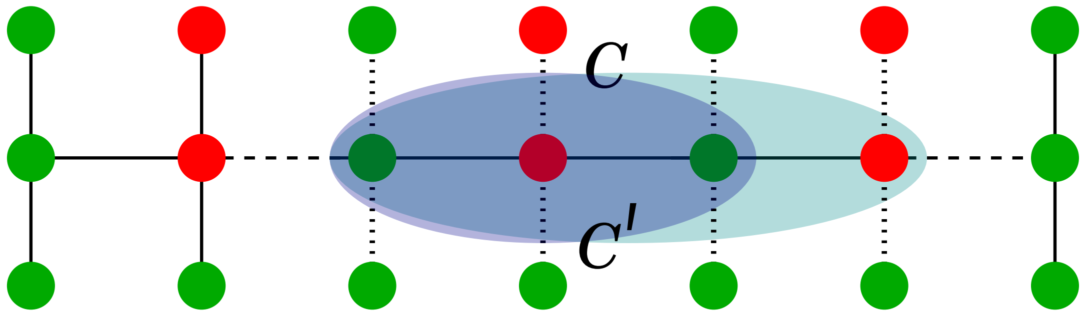
Improving set in a 2-colored graph $G = (V, E)$

- Subset $A \subseteq V$ such that by flipping the colors of nodes in A the potential decreases
- $\text{Imp}(A)$ = improvement in the potential
- Improving ratio: $\text{IR}(A) = \text{Imp}(A)/|A|$

C not minimal







$\text{Imp}(A) = 3 - 1 = 2$

$\text{IR}(A) = 2/1 = 2$

$\text{Imp}(B) = 4$

$\text{IR}(B) = 4/3$

$\text{Imp}(C) = 2$

$\text{IR}(C) = 1/2$

$\text{Imp}(C') = 2$

$\text{IR}(C') = 2/3$

An **error** is always a minimal improving set

Minimal improving set: improving set A such that

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Improving sets

Property 1: on minimal improving sets

G

- $A \subseteq V$ minimal improving set
- $\mathrm{IR}(A) \geq x$
- $\varepsilon < x$

$\Longrightarrow$ for all $v \in A$, $\exists r = O(\log n / \varepsilon)$ and minimal improving set $A' \subseteq \mathcal{N}_r[v] \cap A$ such that $\mathrm{IR}(A') \geq x - \varepsilon$

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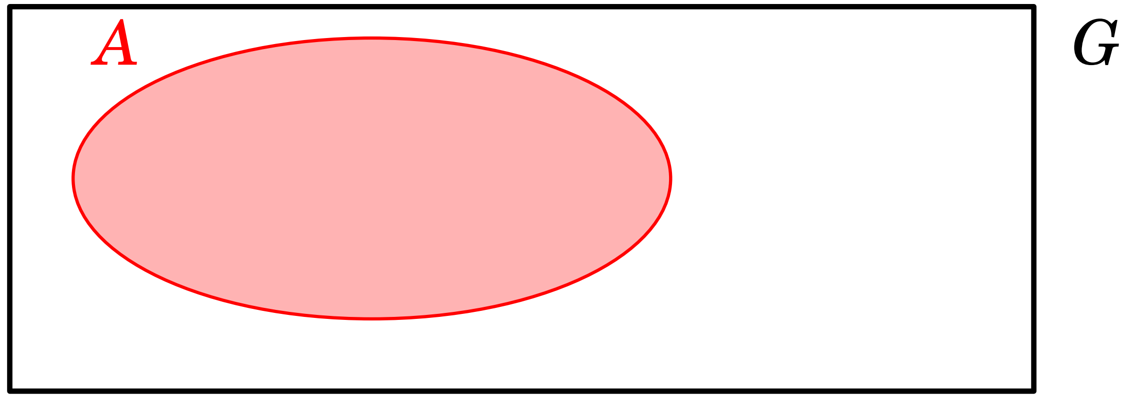

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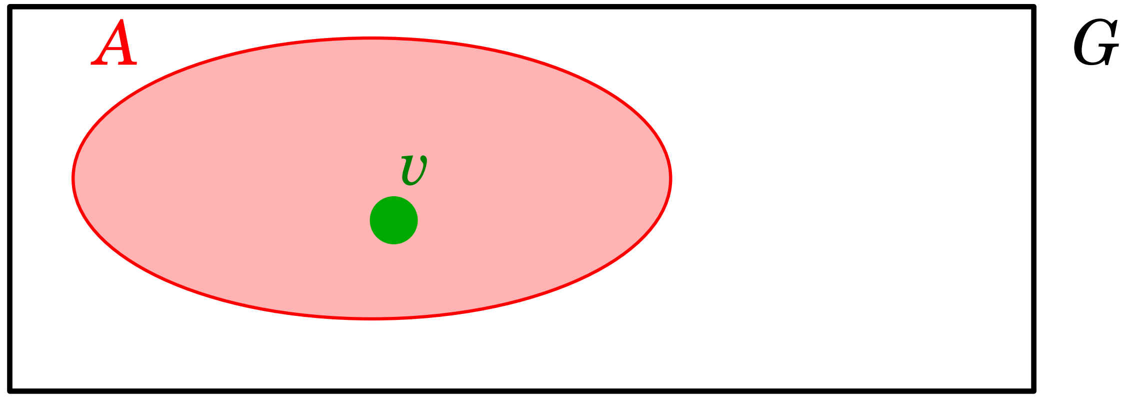

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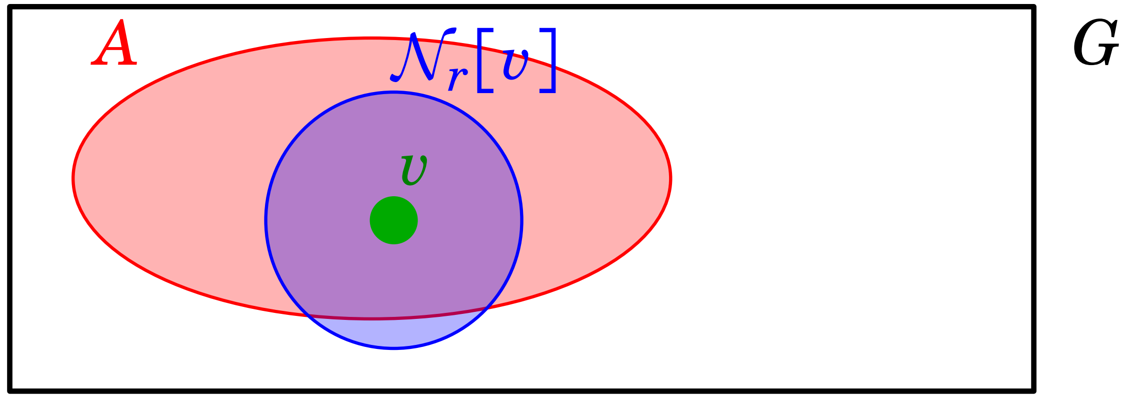

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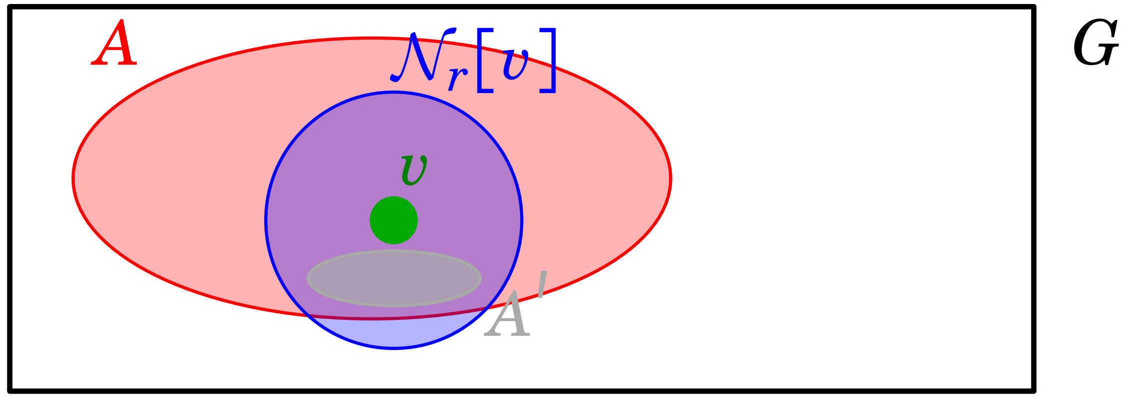

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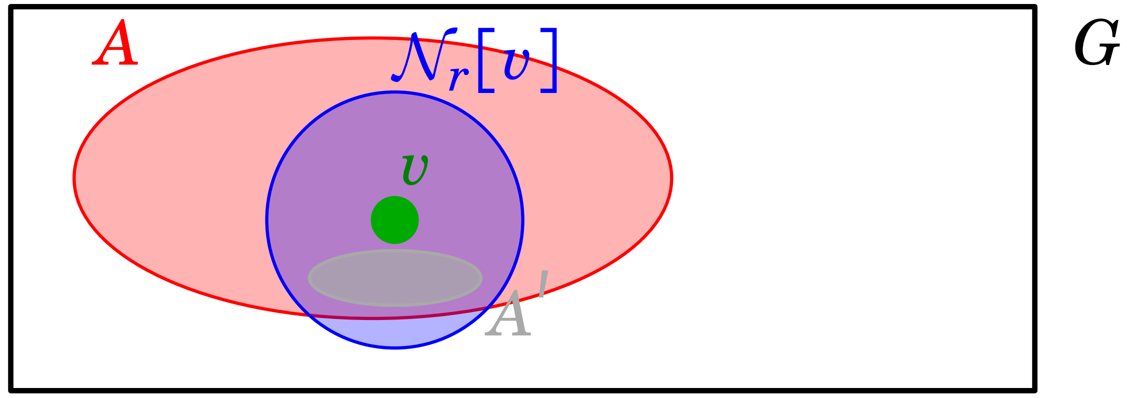

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Sequence of x-improving sets:

- $A_1, \ldots, A_k \subseteq V$
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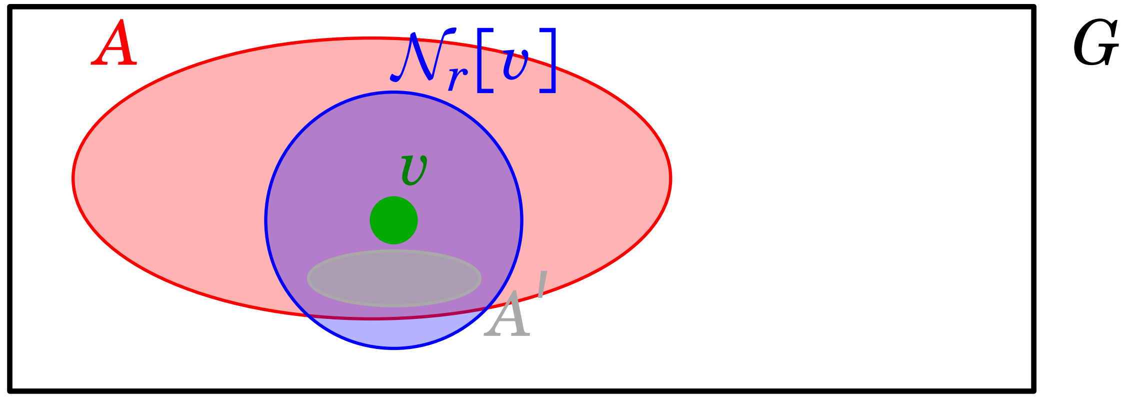

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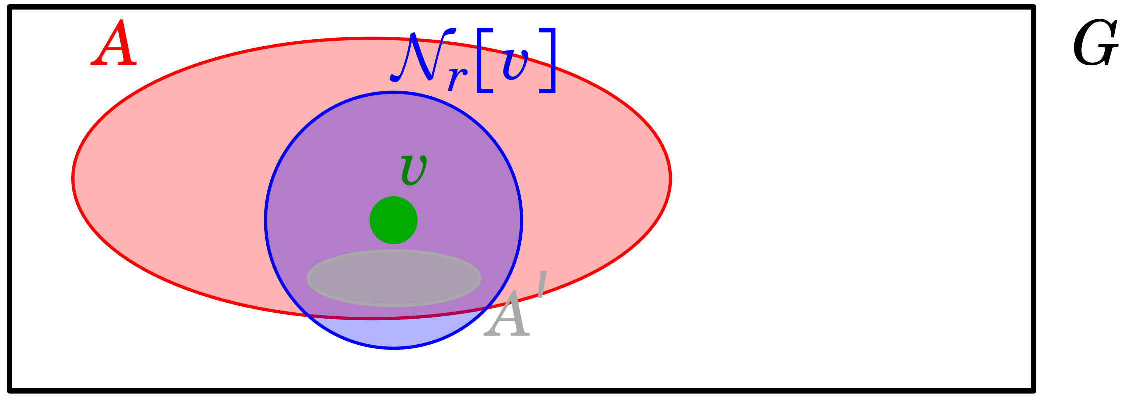

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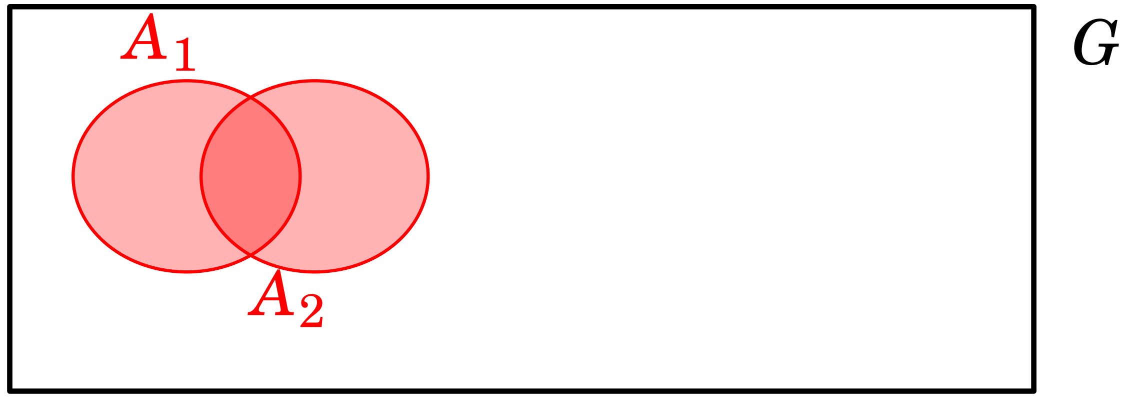

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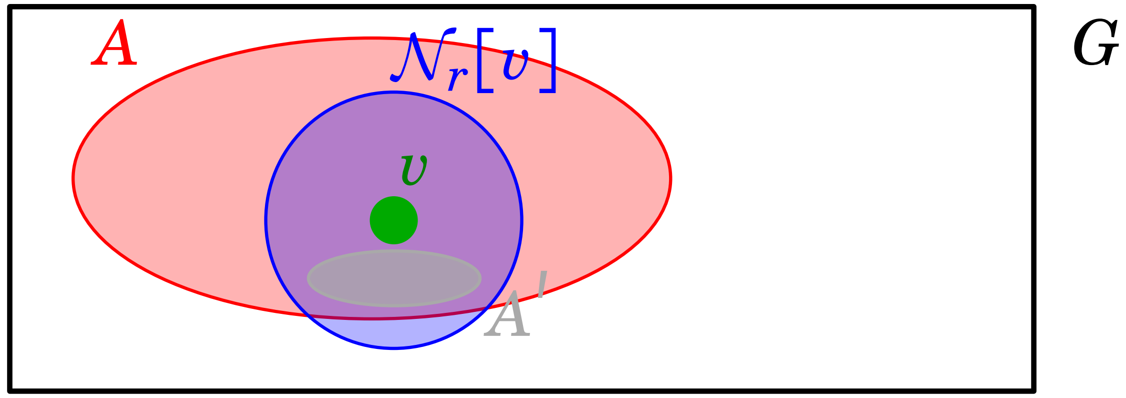

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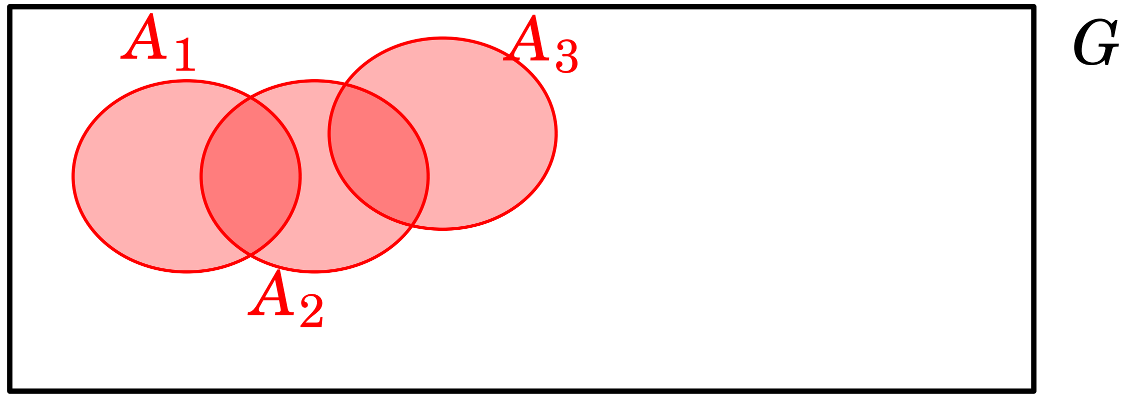

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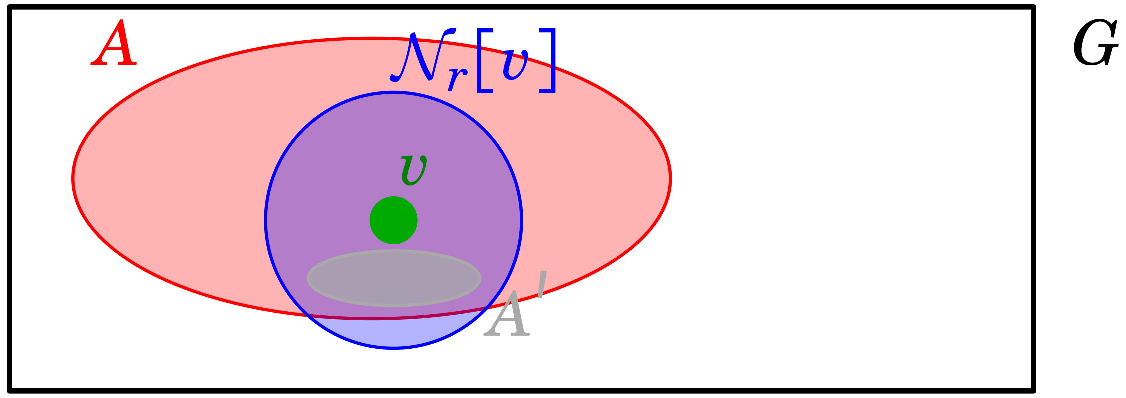

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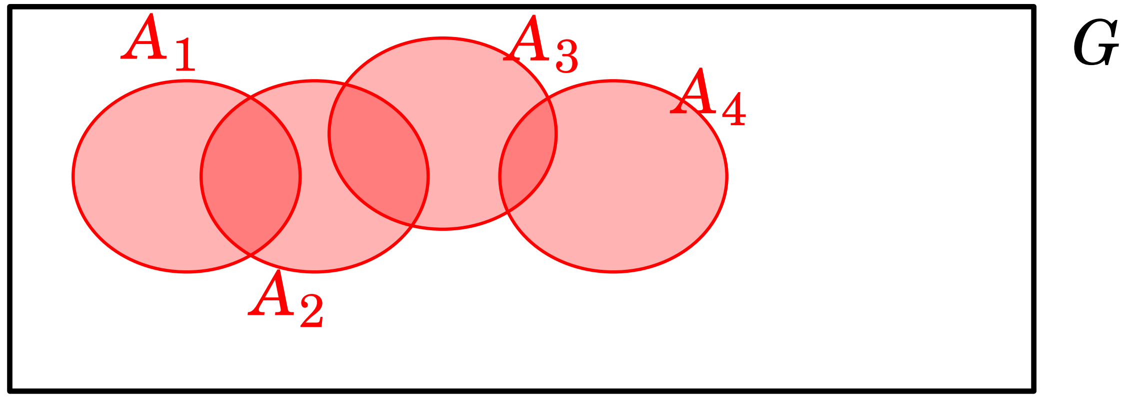

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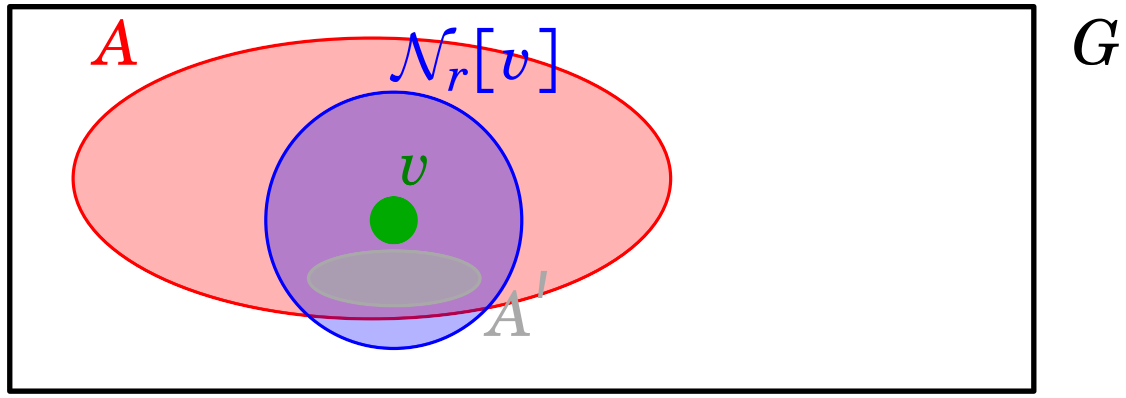

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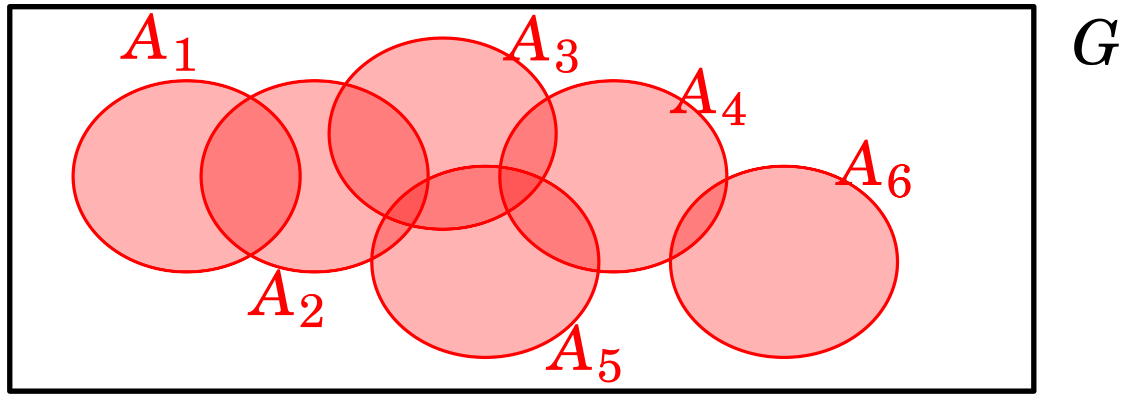

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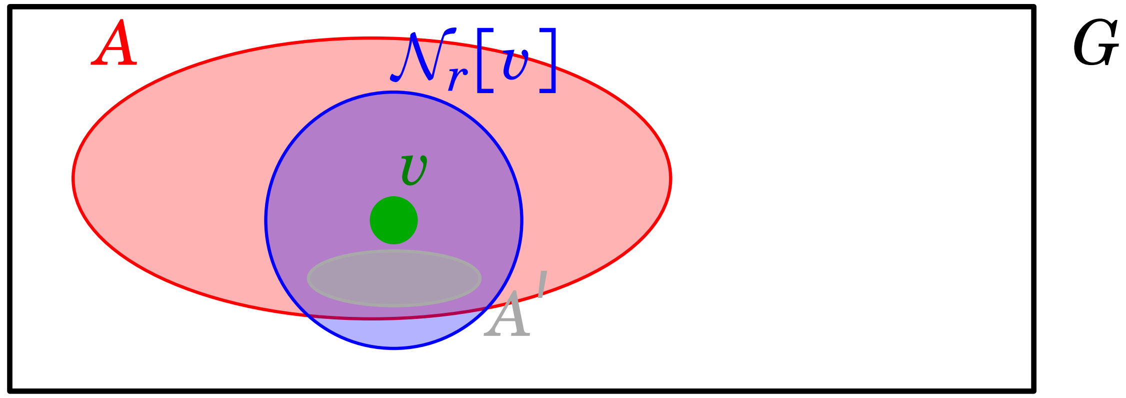

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Property 2: on sequences of x-improving sets

- $A_1, \ldots, A_k \subseteq V$ sequence of x-improving sets
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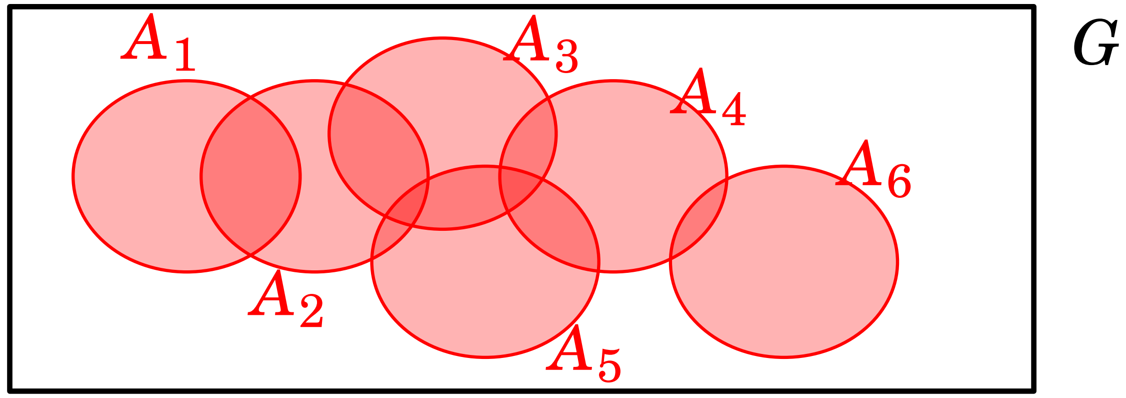
- $\varepsilon < x$
- $A = \cup_i A_i$



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Improving sets

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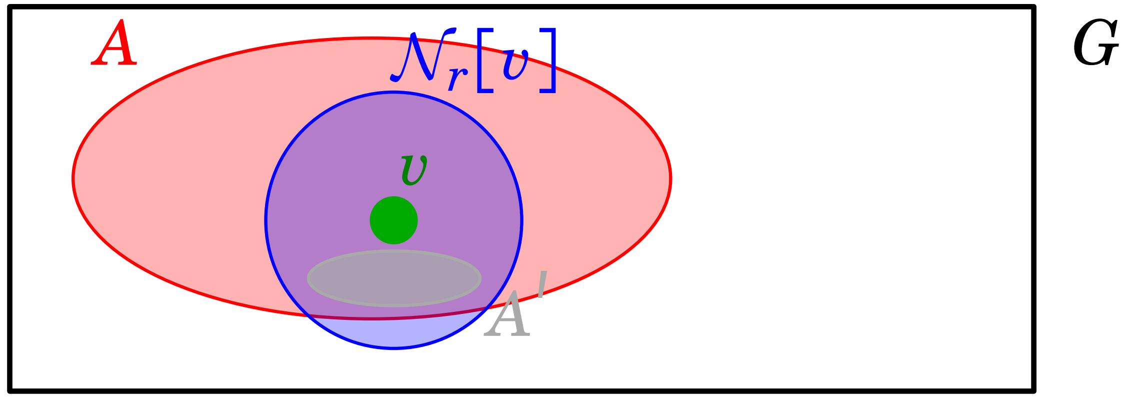

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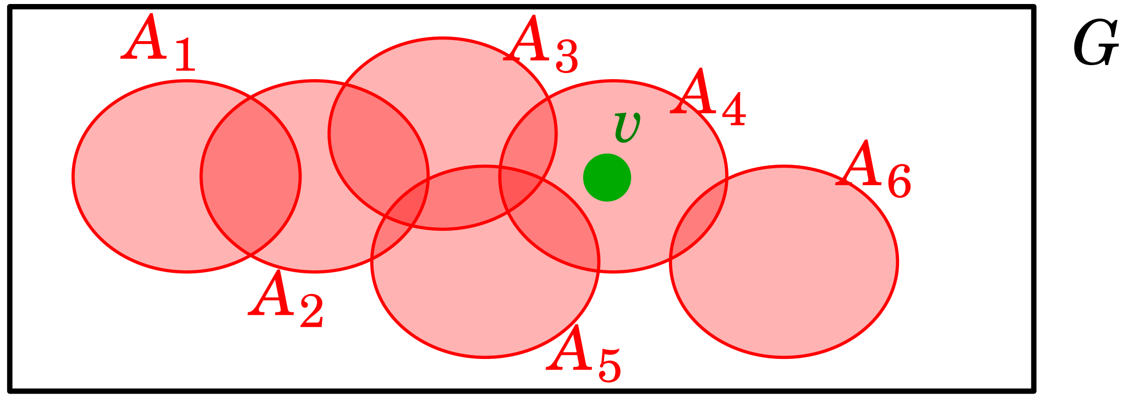

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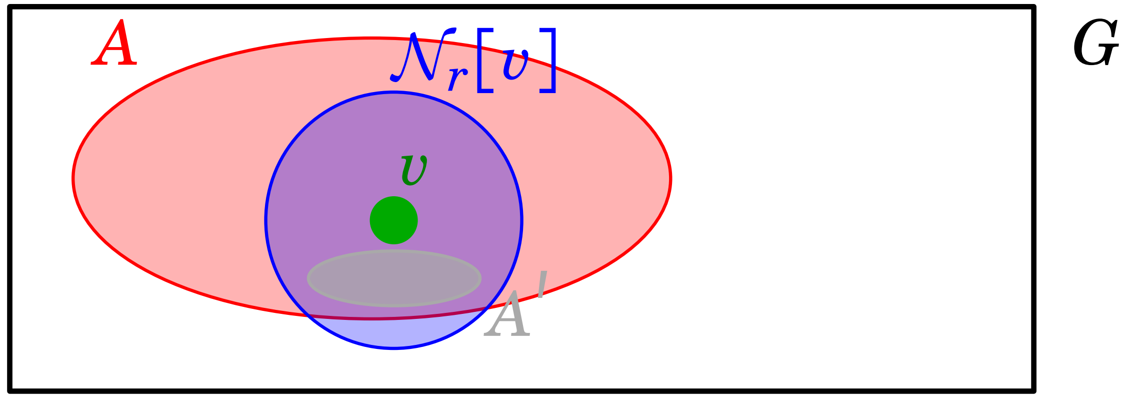

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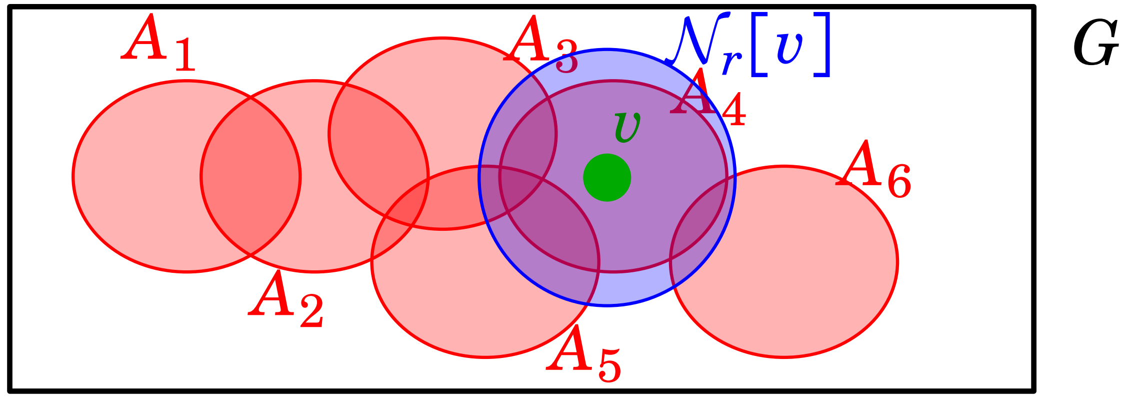

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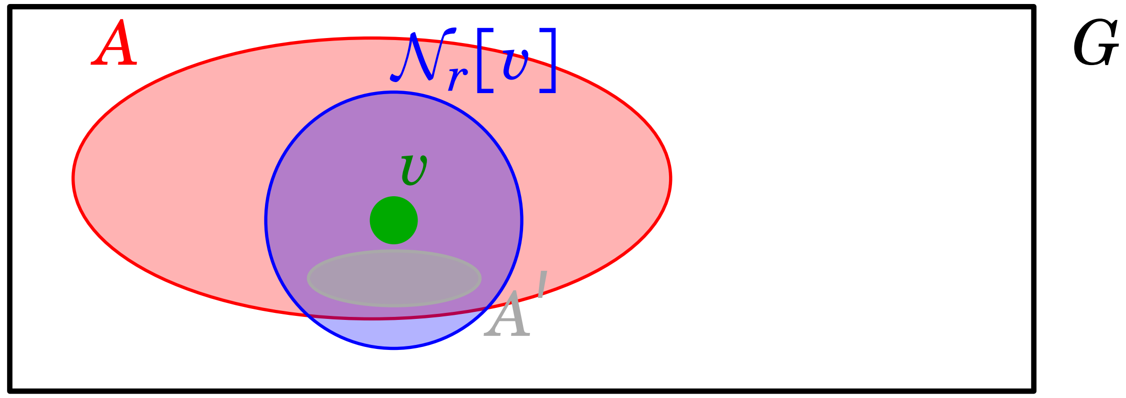

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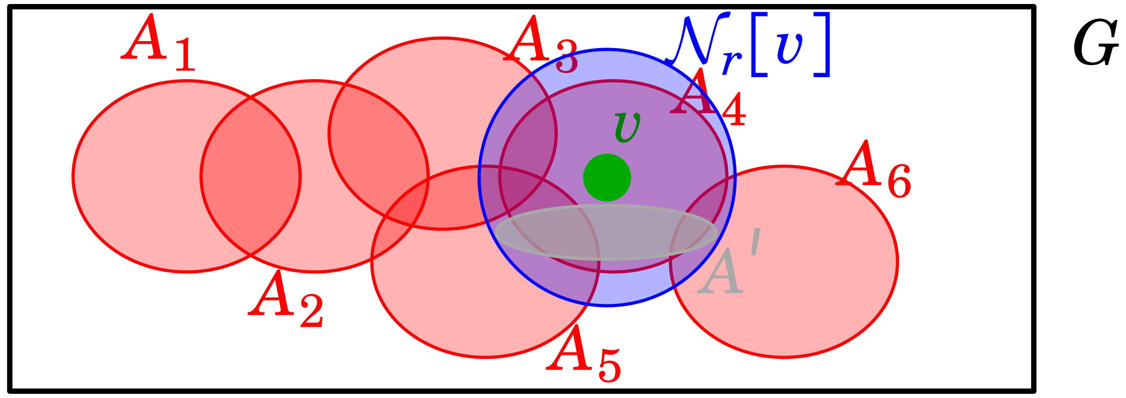

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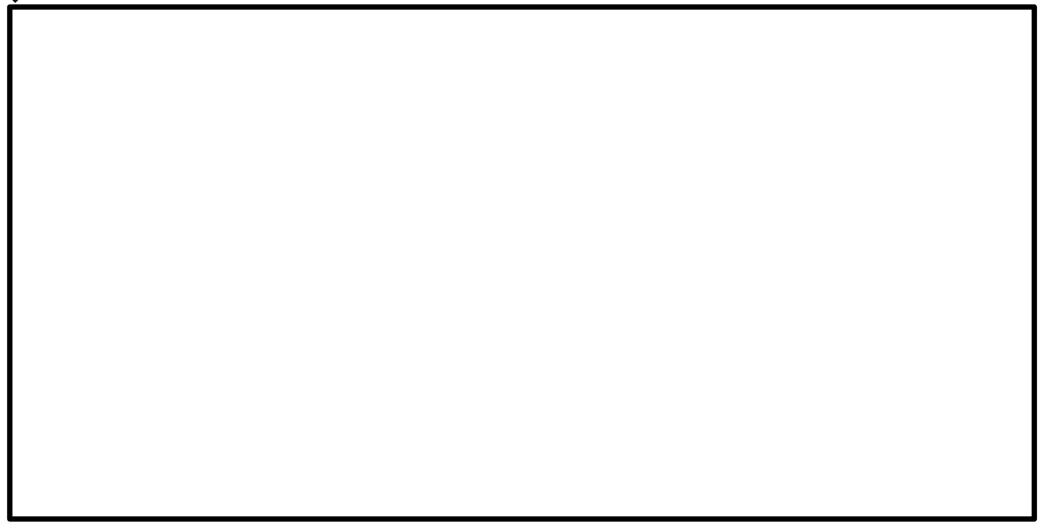
- Set $\lambda_1 = 1/4$ (*initial IR*), $\varepsilon = \lambda/(2000 \log n)$ (*Properties* 1,2), $\alpha = \Theta(\varepsilon^2/\log^2 n)$ ($\mathcal{MPX}$)
- Start with a random coloring assignment



G

The algorithm

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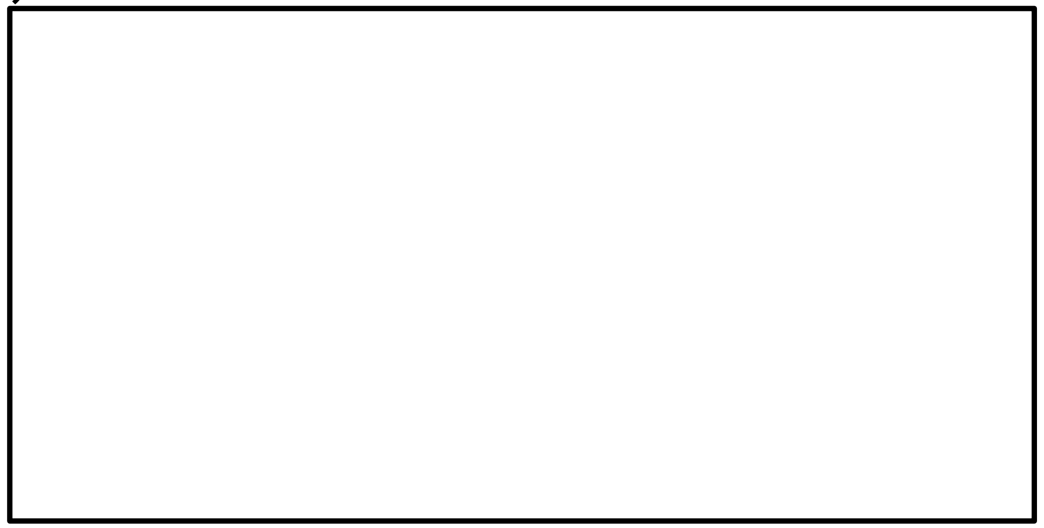

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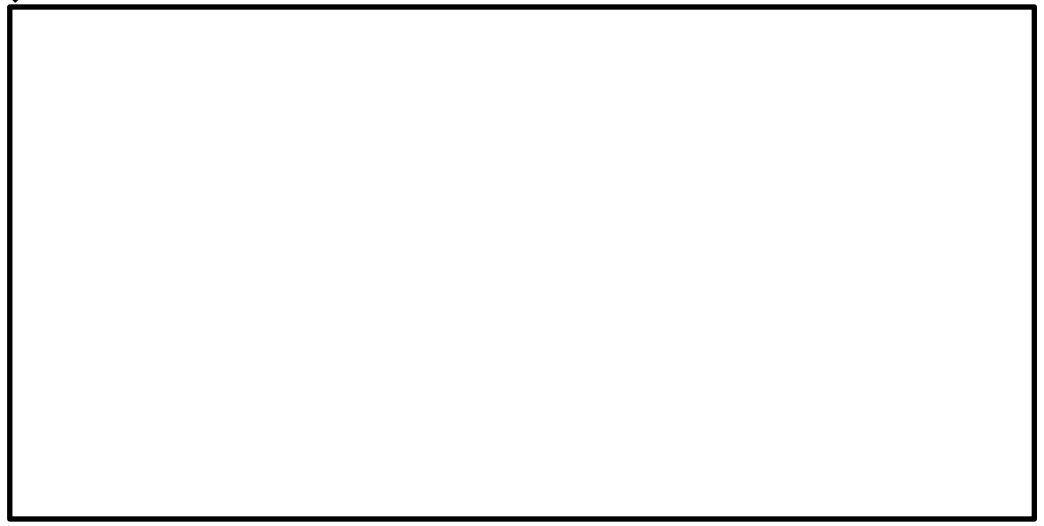

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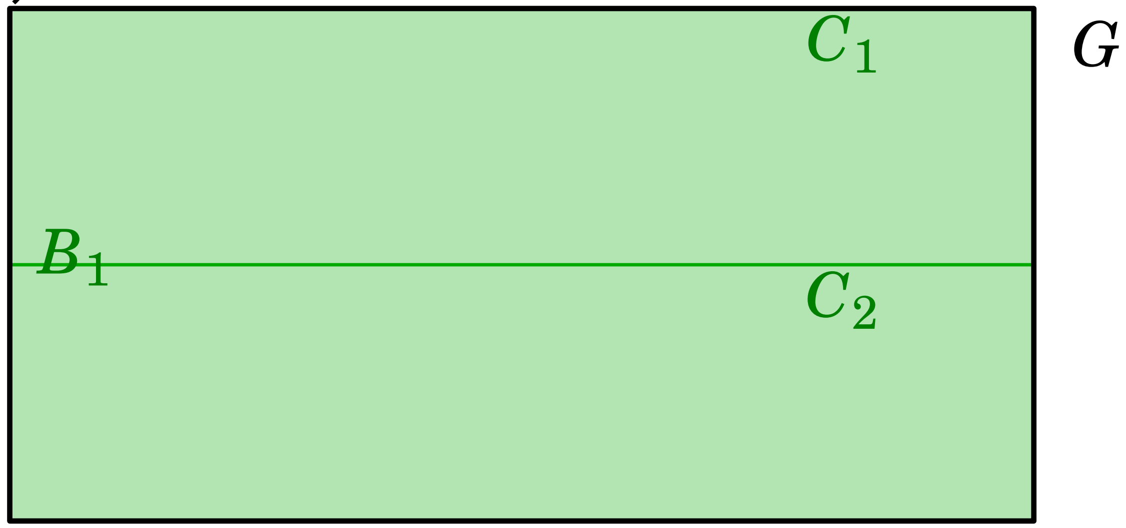

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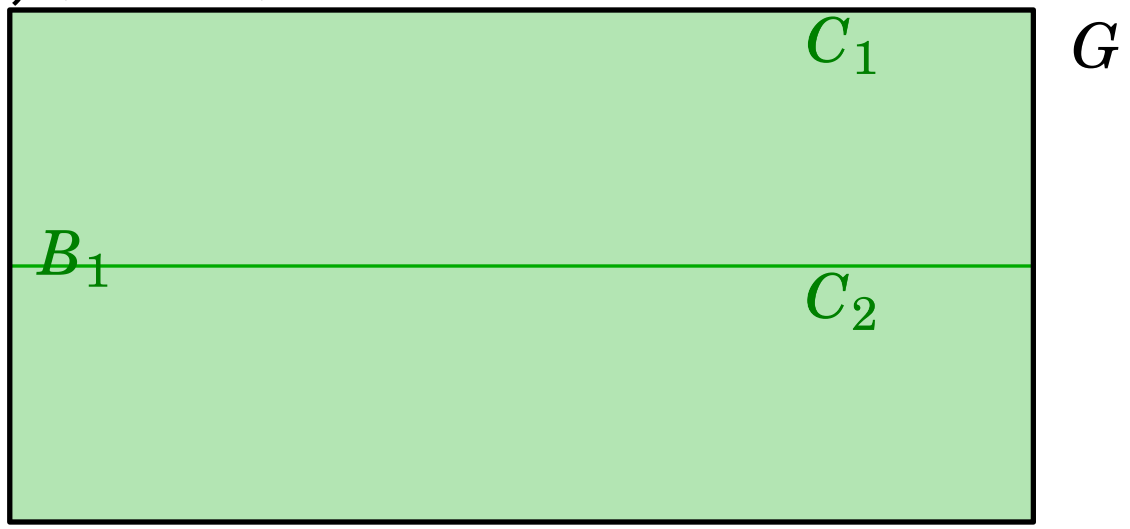

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  - Find maximal sequence σ of λ_i-improving sets of diam. $O(\log n/\varepsilon)$ inside $G[\mathcal{C}]$ that do not contain border nodes

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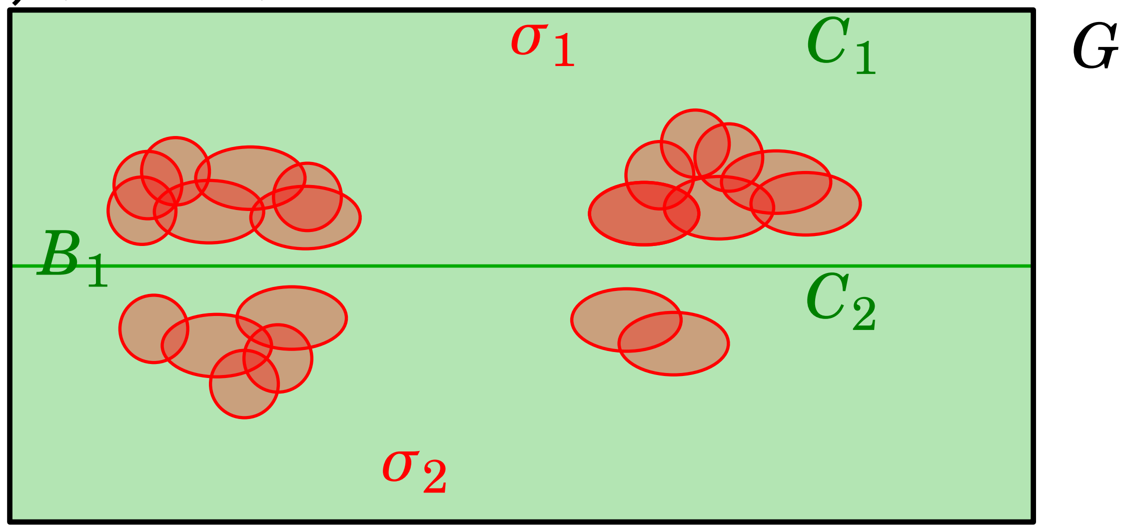

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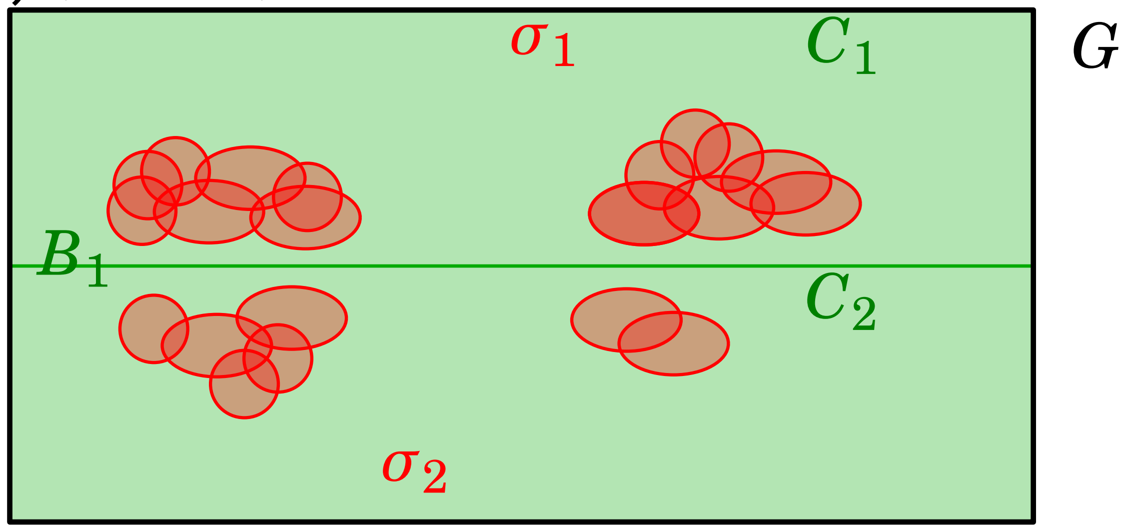

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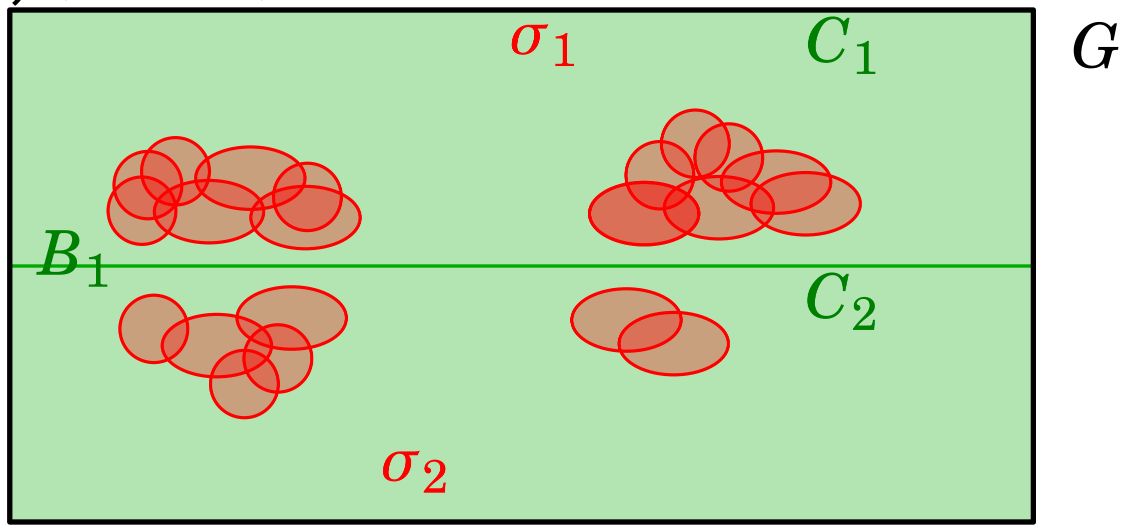
  - $\lambda_{i+1} = \lambda_i + 10\varepsilon$



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PHASE

- Run $\mathcal{MPX}$ to get an (α, d)-decomposition
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- Flip all sets in σ, in order
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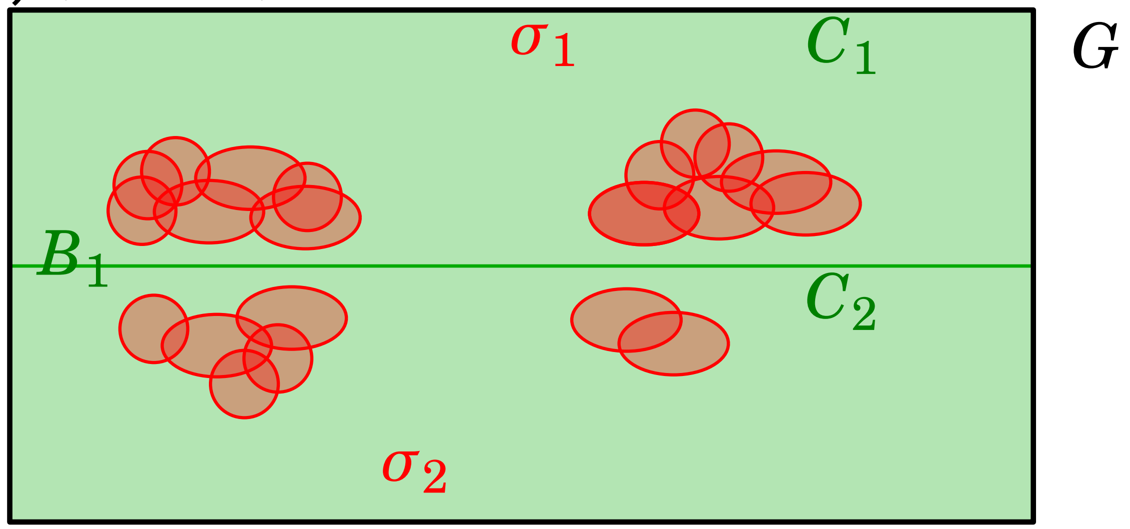

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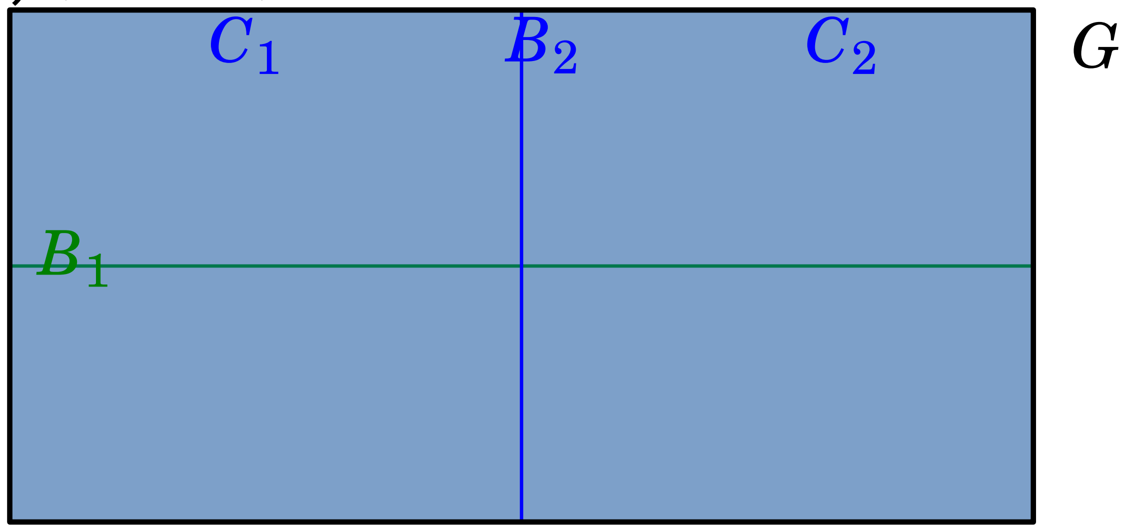

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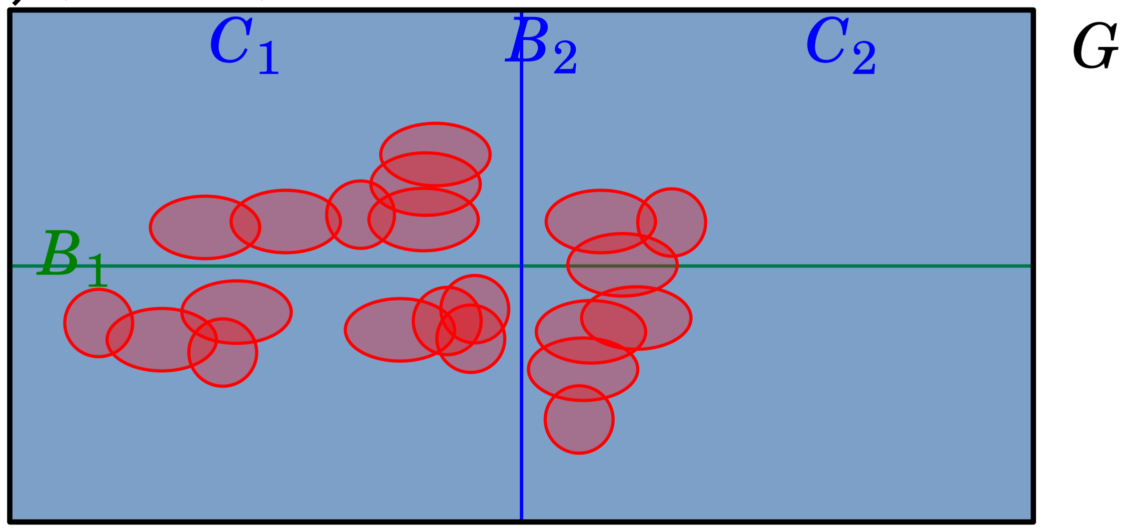

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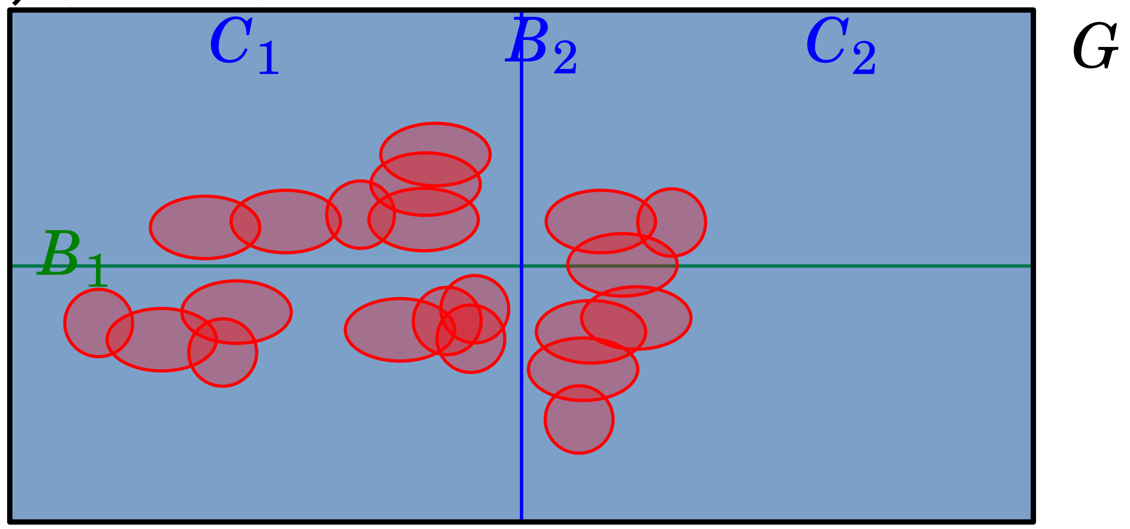

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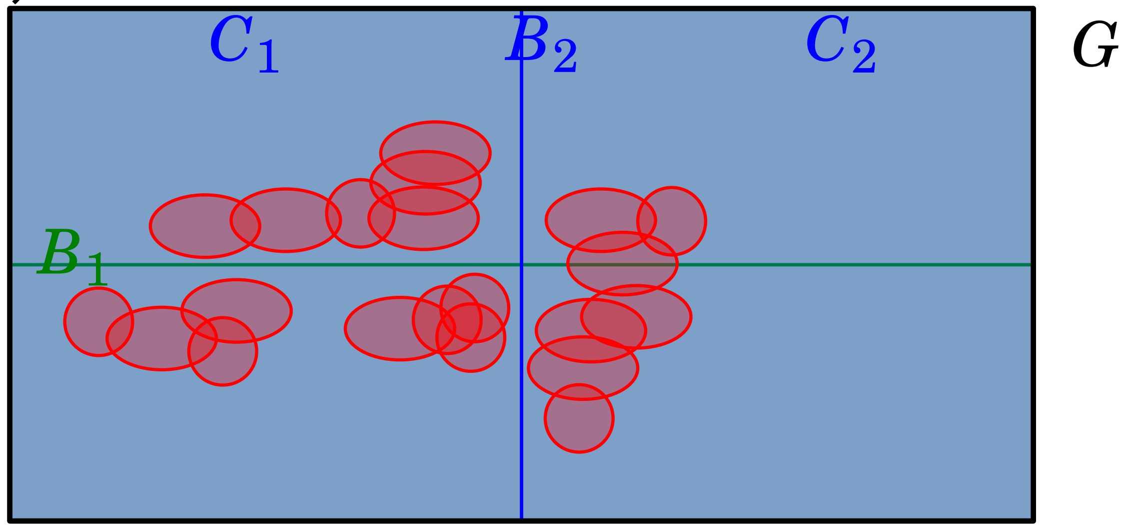

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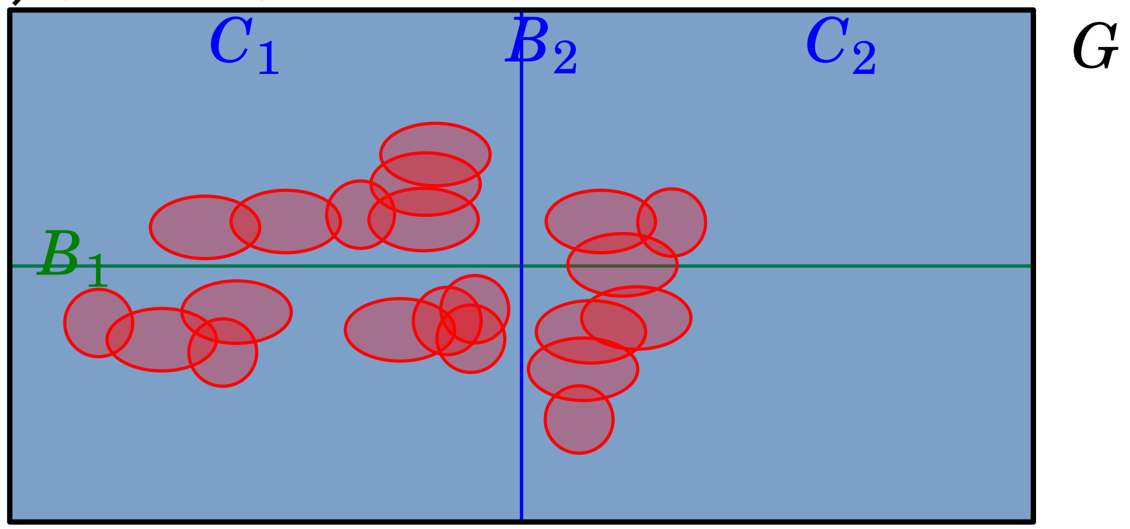

Claim 1: *The running time of the algorithm is* $O(d \cdot \log n) = O(\log^2 n/\alpha) = O(\log^6 n)$

Proof 1: $O(\log n)$ phases. Each phase costs $O(d)$. By $\mathcal{MPX}$, $d = O(\log n/\alpha)$. By def. $\alpha = O(1/\log^4 n)$.

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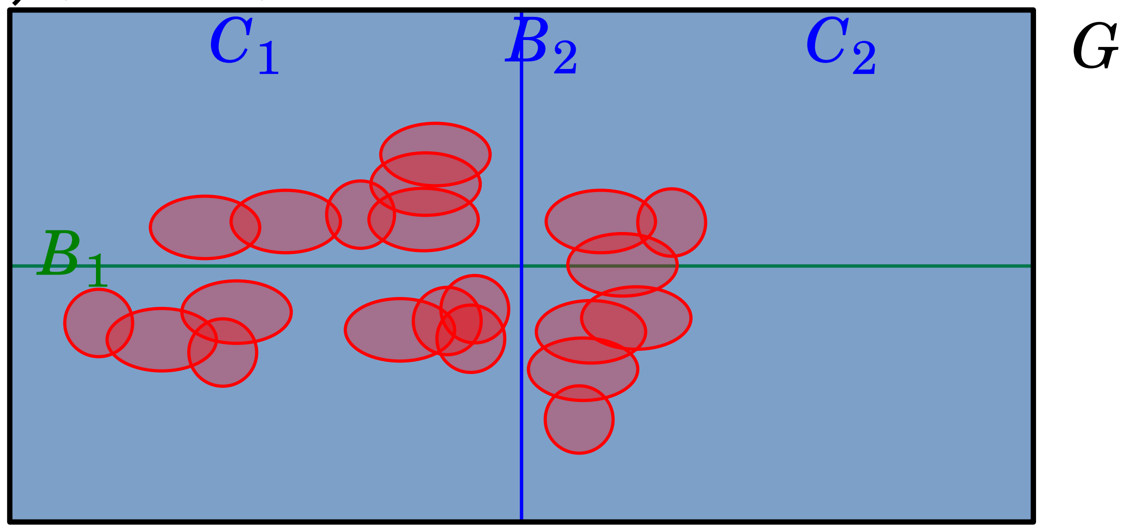

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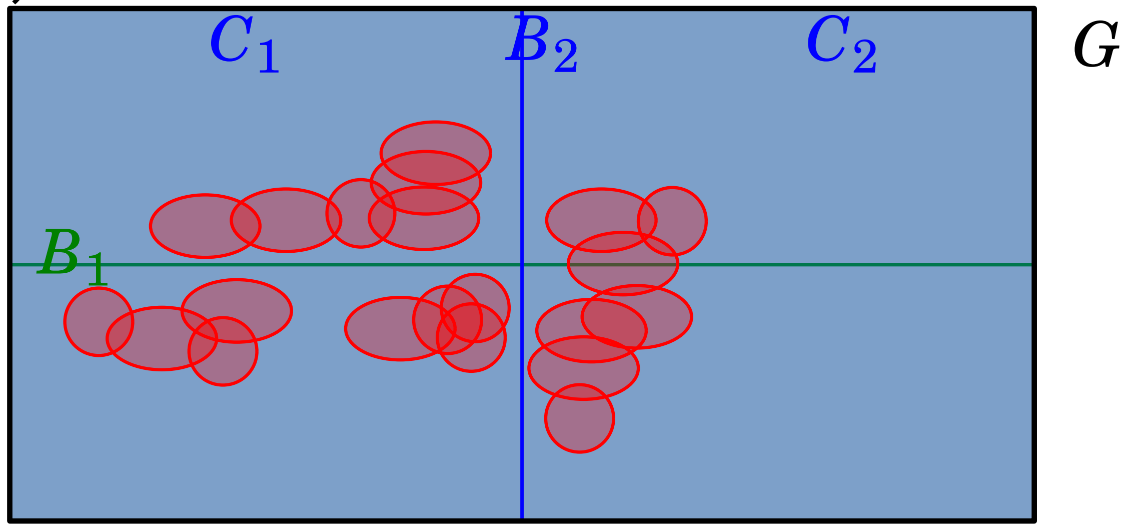

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Property 1 + Property 2

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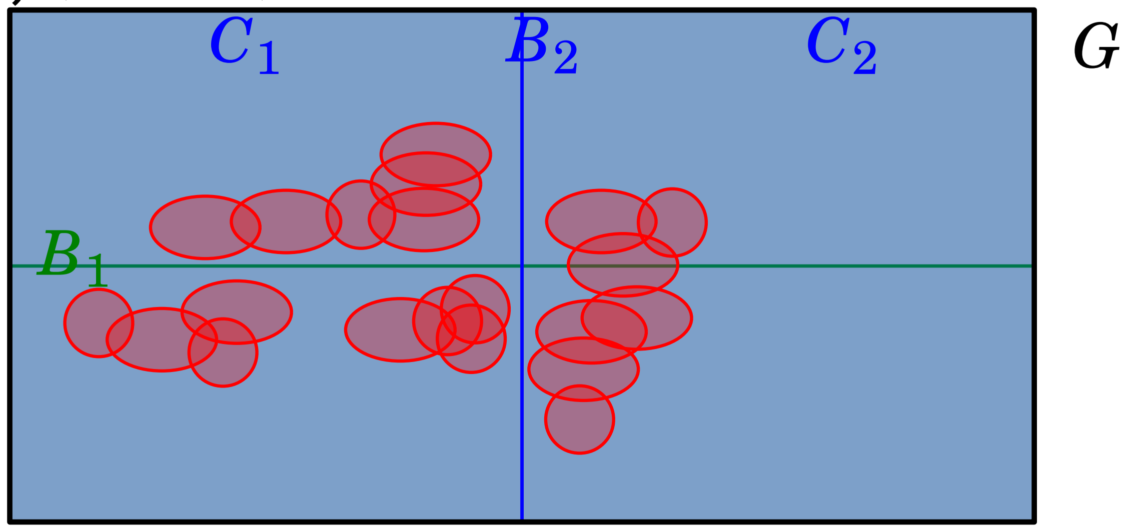

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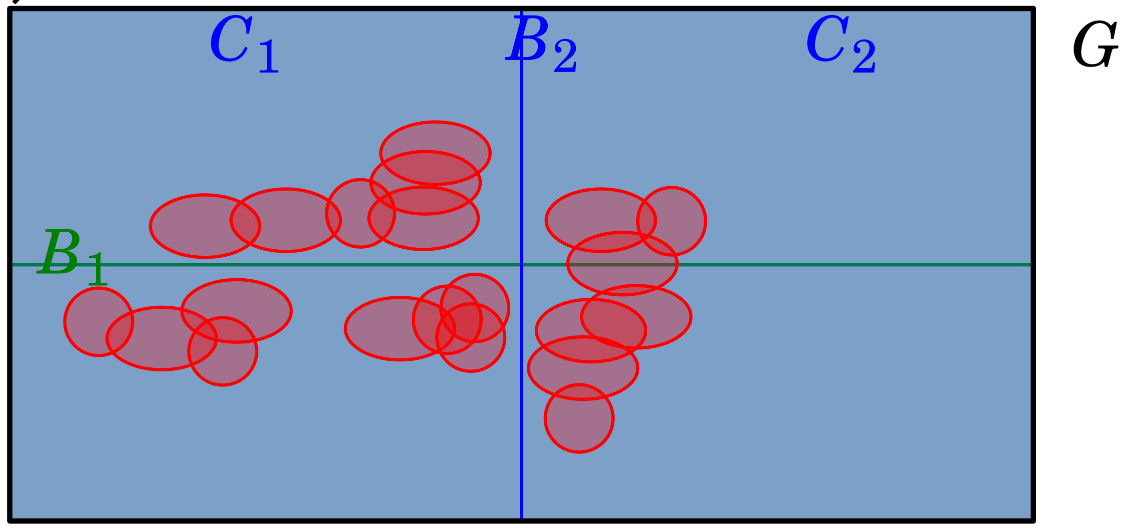

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Local potential problems

- $r > 0$, $\Delta > 0$
- $\mathcal{C}$ list of valid neighborhoods of radius $\boldsymbol{r}$ and max degree $\boldsymbol{\Delta}$

Local potential problems

- $r > 0$, $\Delta > 0$
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locally optimal cut with
$\boldsymbol{\Delta = 3}$ (r = 1)

valid neighborhoods

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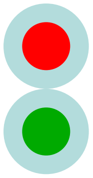

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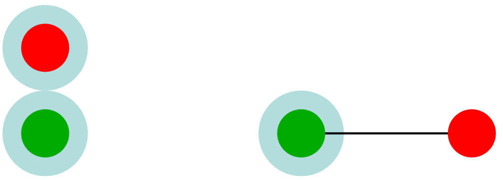

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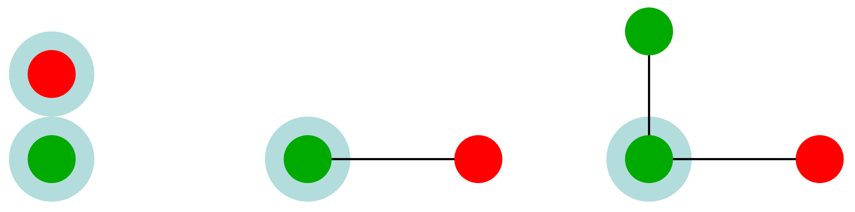

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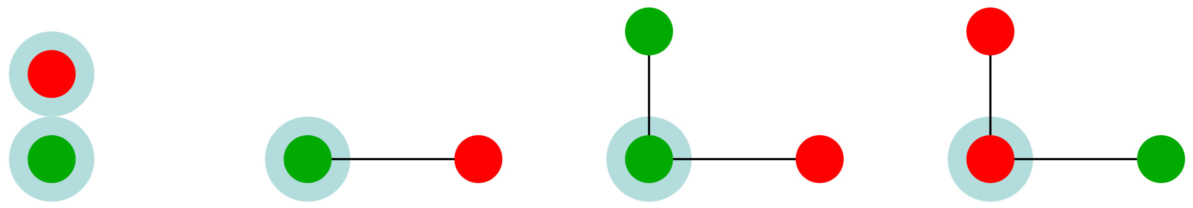

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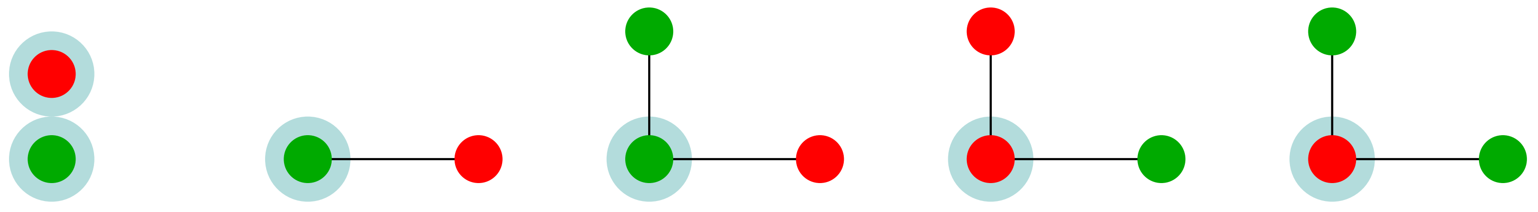

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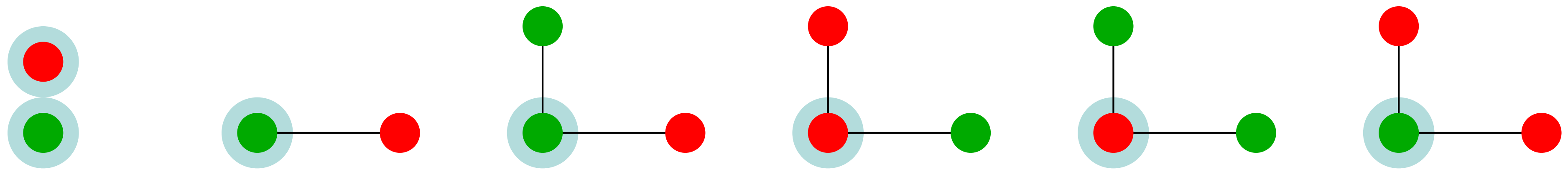

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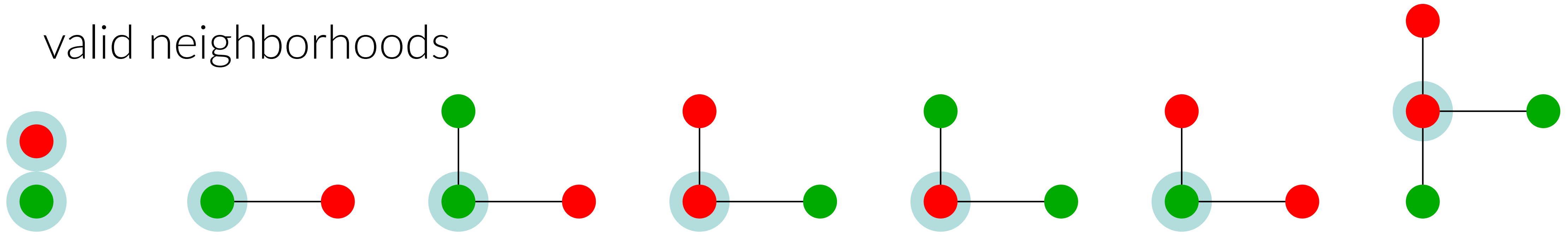

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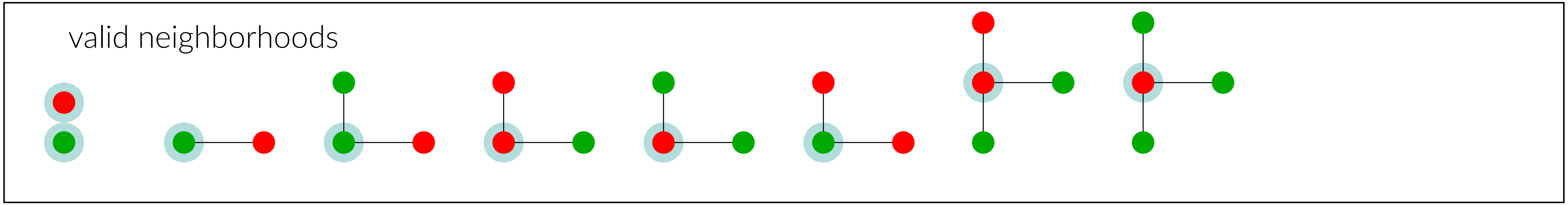

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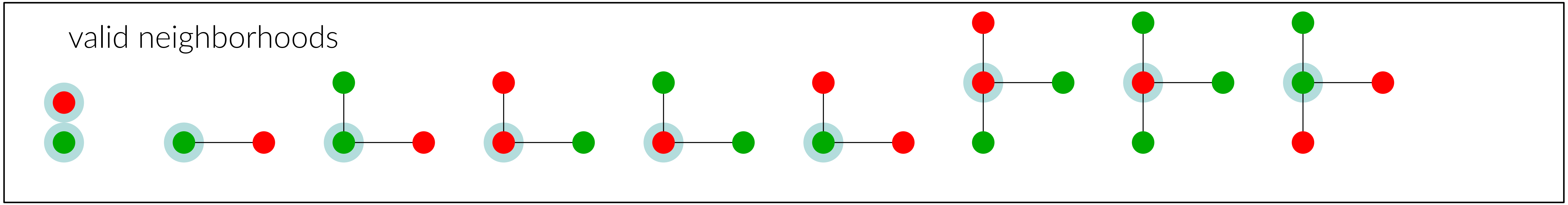

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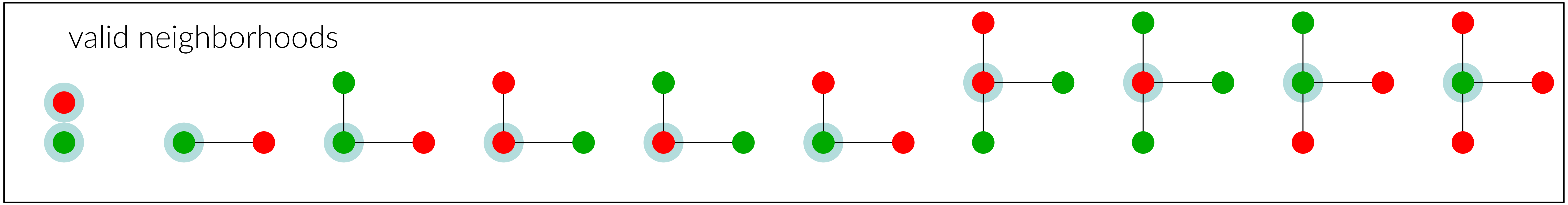

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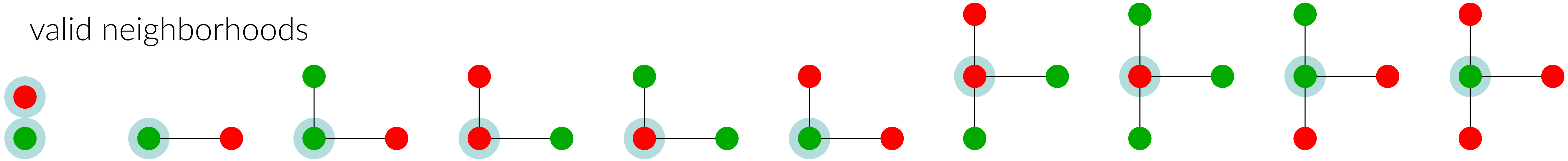

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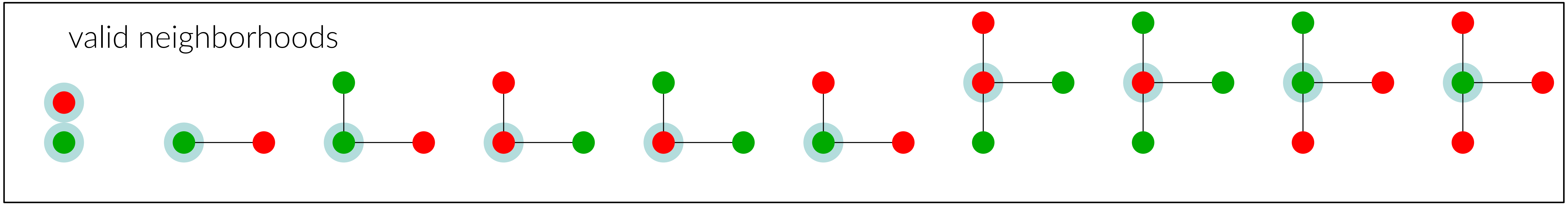

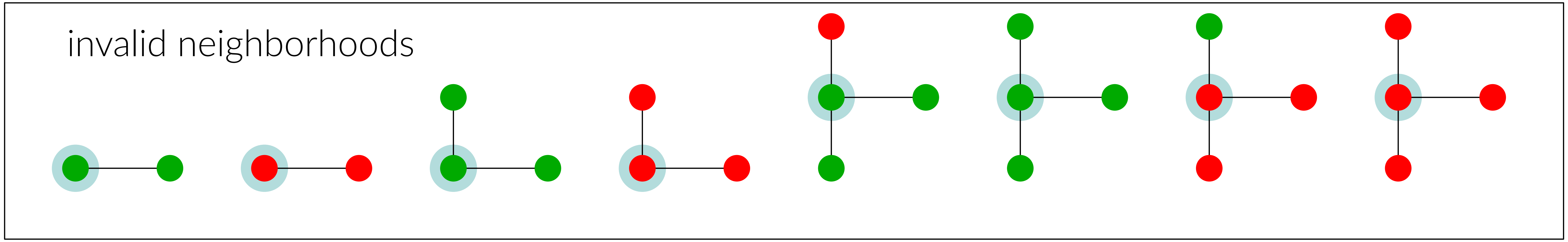

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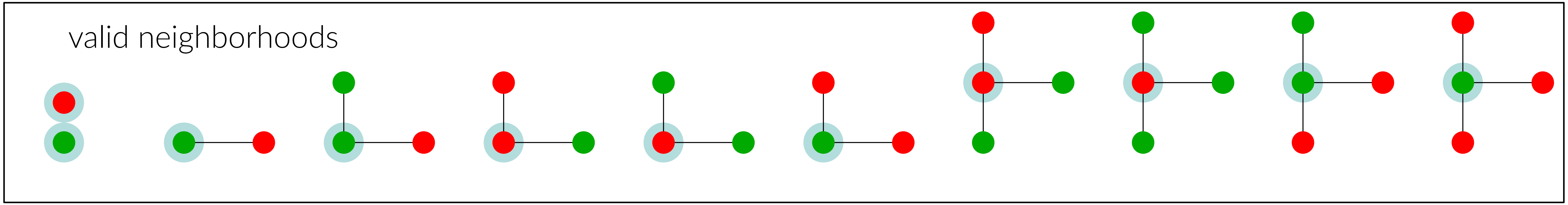

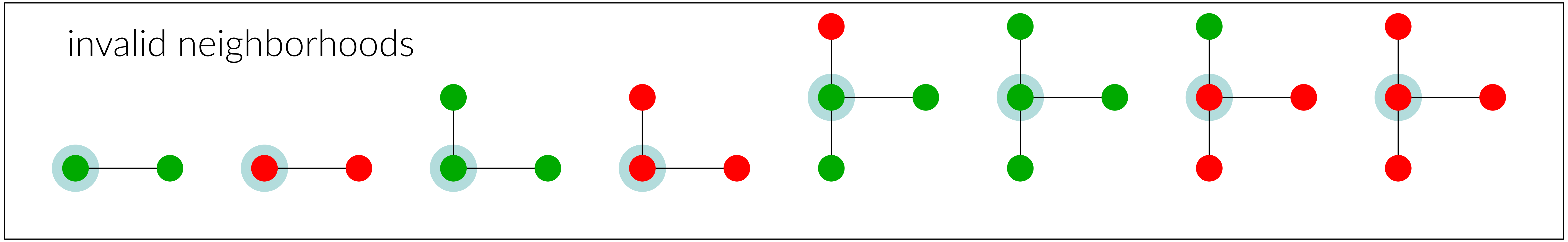

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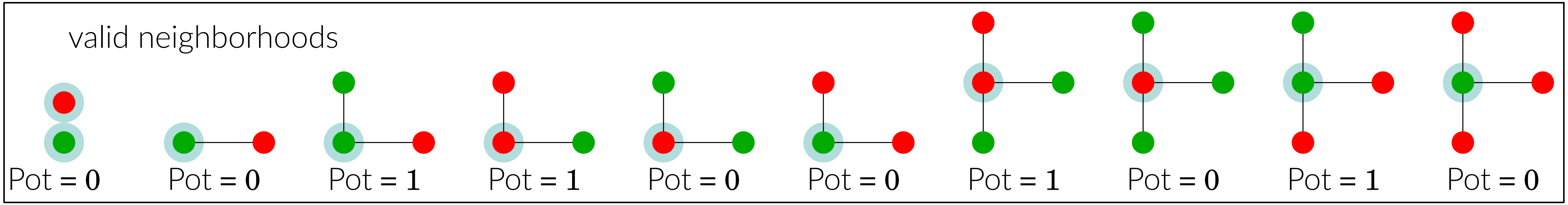

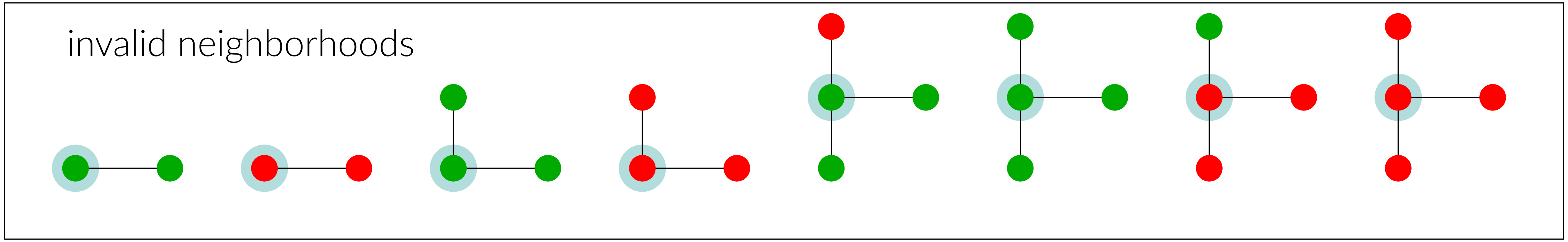

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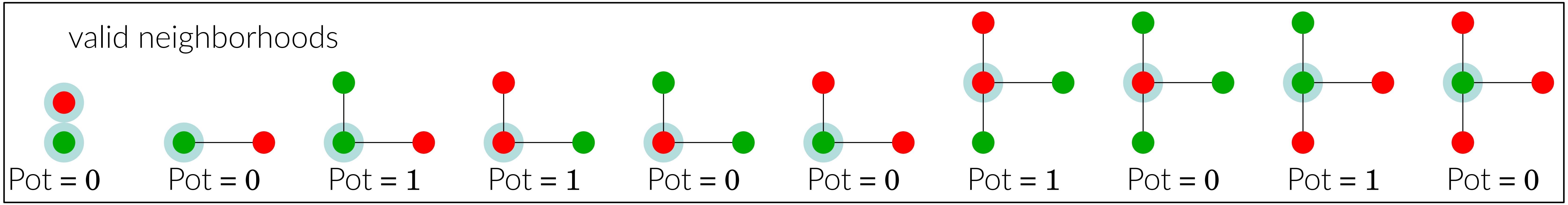

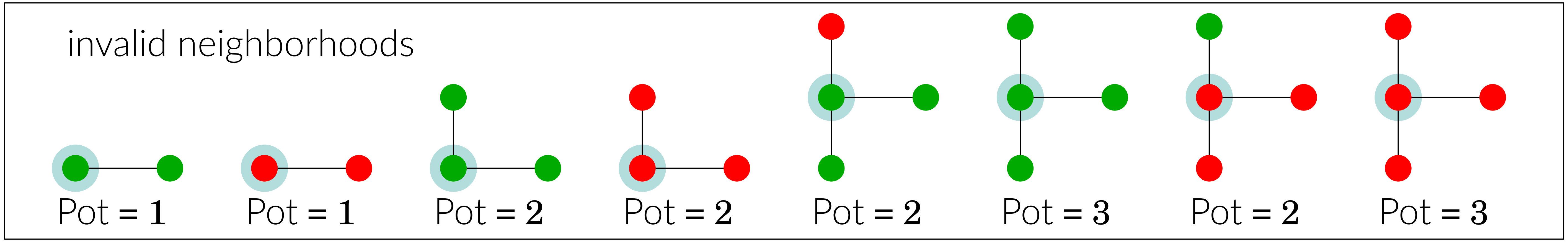

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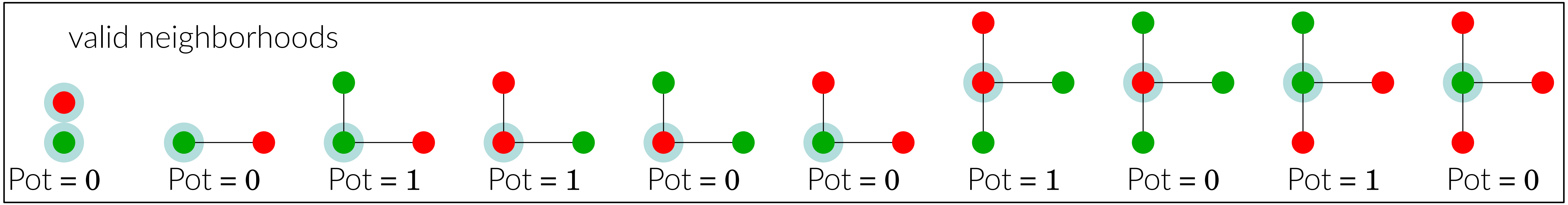

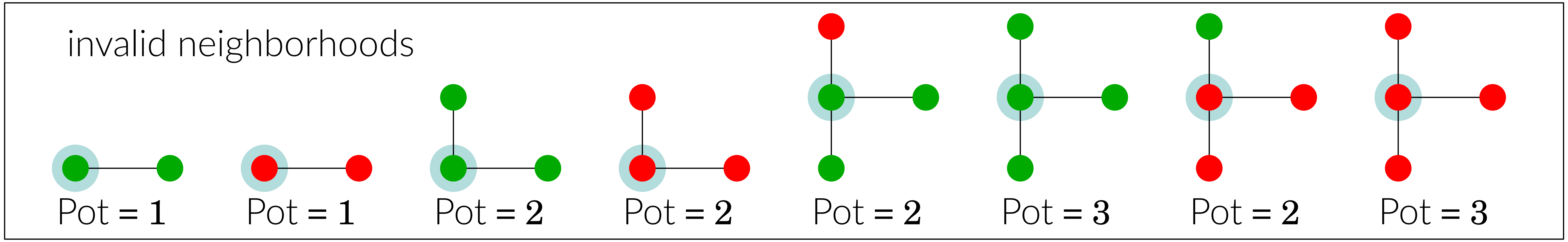

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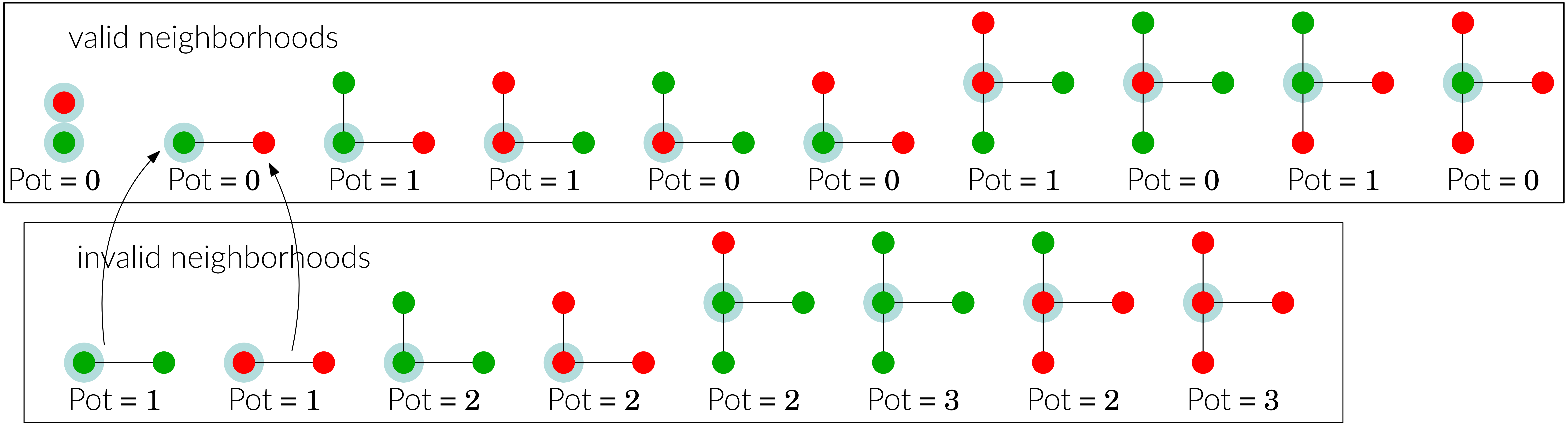

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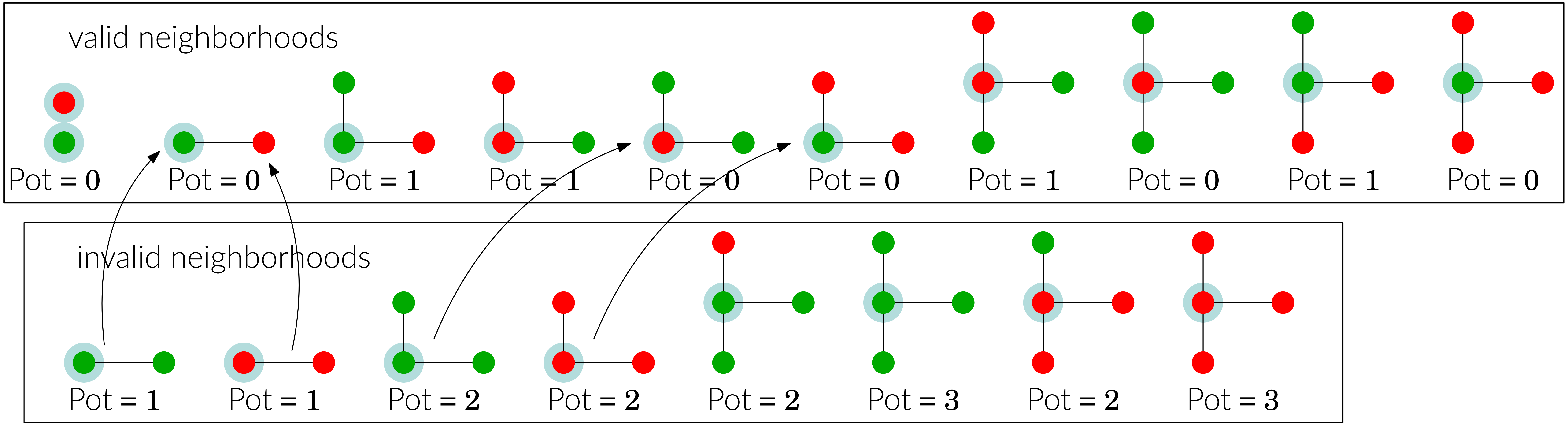

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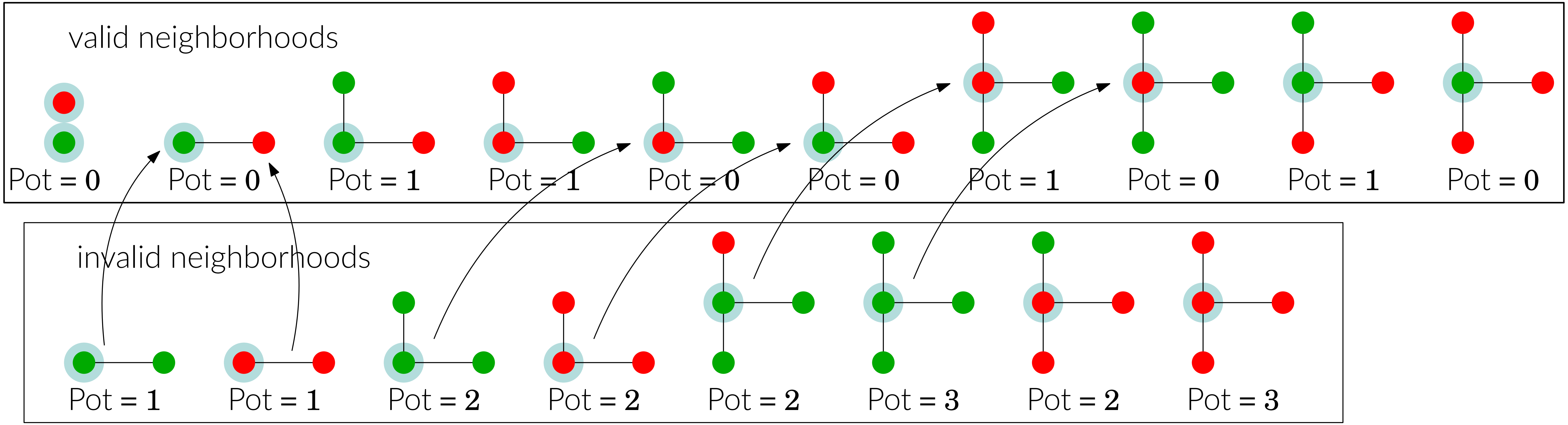

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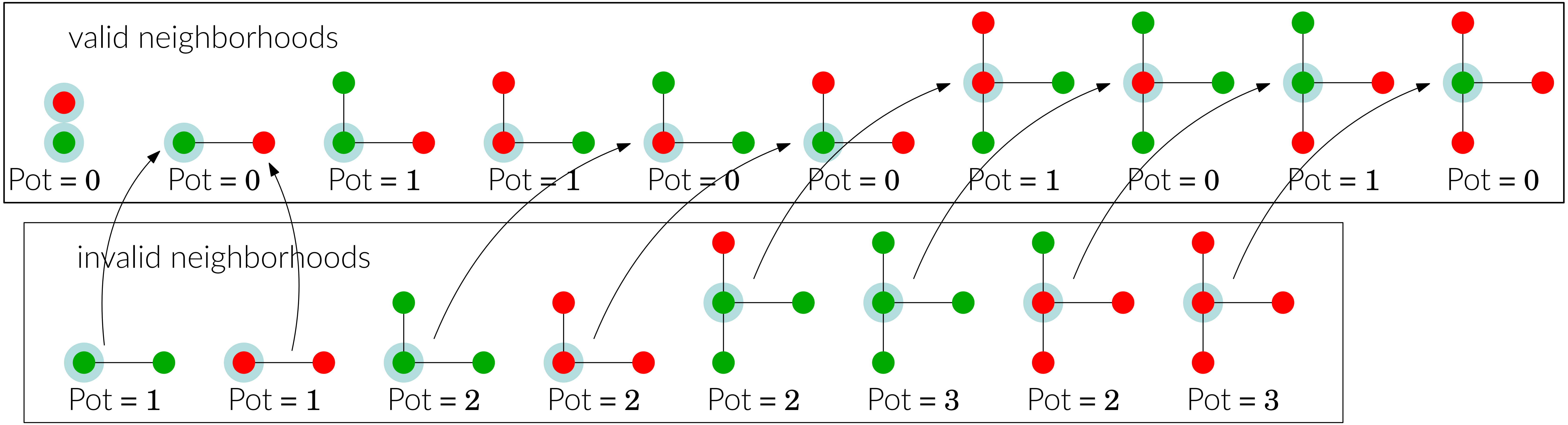

Recap & conclusions

Theorem:

For any local potential problem Π, *there exists a randomized LOCAL algorithm that solves* Π *with high probability in time* $O(\log^6 n)$. *The latter can be derandomized into a deterministic LOCAL algorithm that solves* Π *in time* $O(\log^8 n \text{ poly}(\log\log n))$.

Recap & conclusions

Theorem:

For any local potential problem Π, there exists a randomized LOCAL algorithm that solves Π with high probability in time $O(\log^6 n)$. The latter can be derandomized into a deterministic LOCAL algorithm that solves Π in time $O(\log^8 n$ poly$(\log\log n))$.

Lower bound:

- $\Omega(\log n)$-rounds in deterministic LOCAL (in *bounded-degree* trees)
- $\Omega(\log\log n)$-rounds in randomized LOCAL (in *bounded-degree* trees)

Recap & conclusions

Theorem:

For any local potential problem Π, there exists a randomized LOCAL algorithm that solves Π with high probability in time $O(\log^6 n)$. The latter can be derandomized into a deterministic LOCAL algorithm that solves Π in time $O(\log^8 n \text{ poly}(\log\log n))$.

Lower bound:

- $\Omega(\log n)$-rounds in deterministic LOCAL (in *bounded-degree* trees)
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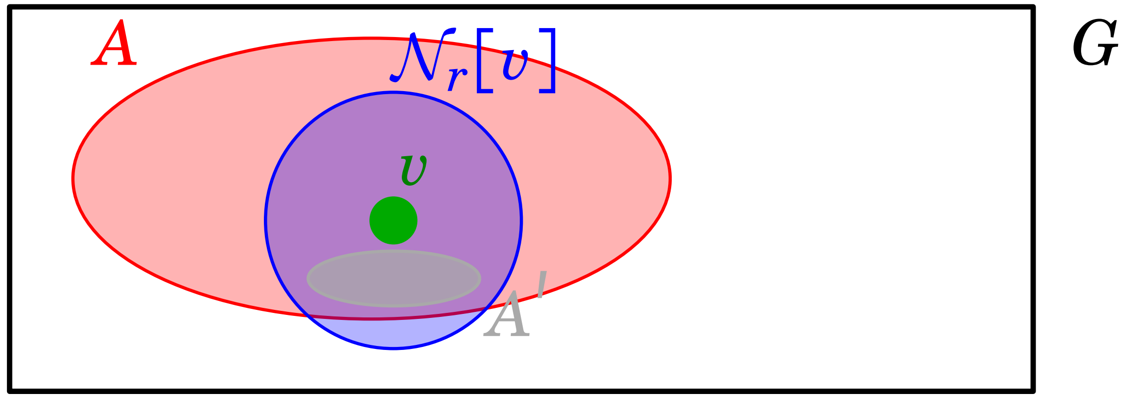
THANKS

Proof of Property 1

Property 1: on minimal improving sets



- $A \subseteq V$ minimal improving set
- $\mathrm{IR}(A) \geq x$
- $\varepsilon < x$

$\implies$ for all $v \in A$, $\exists r = O(\log n/\varepsilon)$ and minimal improving set $A' \subseteq \mathcal{N}_r[v] \cap A$ such that $\mathrm{IR}(A') \geq x - \varepsilon$

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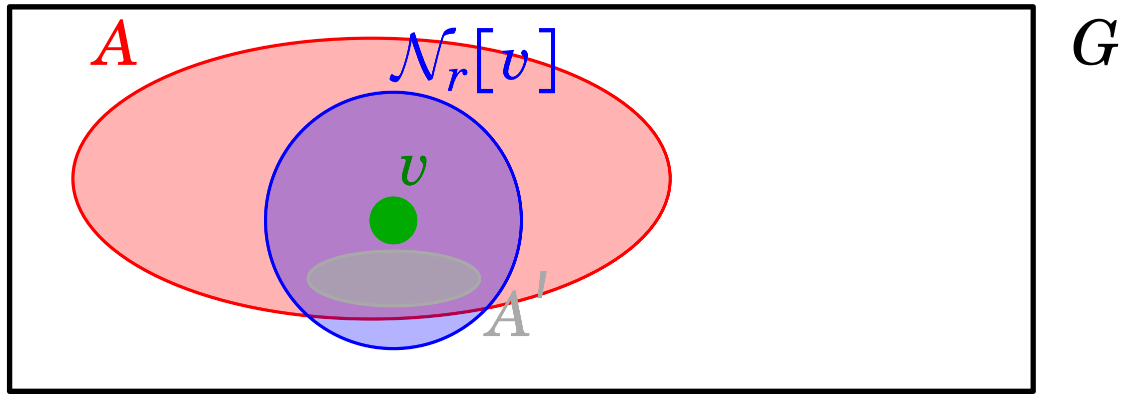

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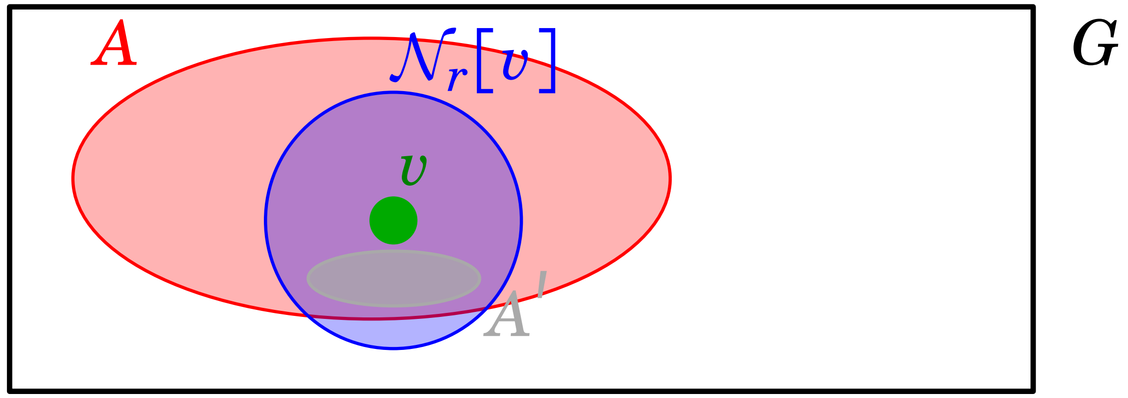

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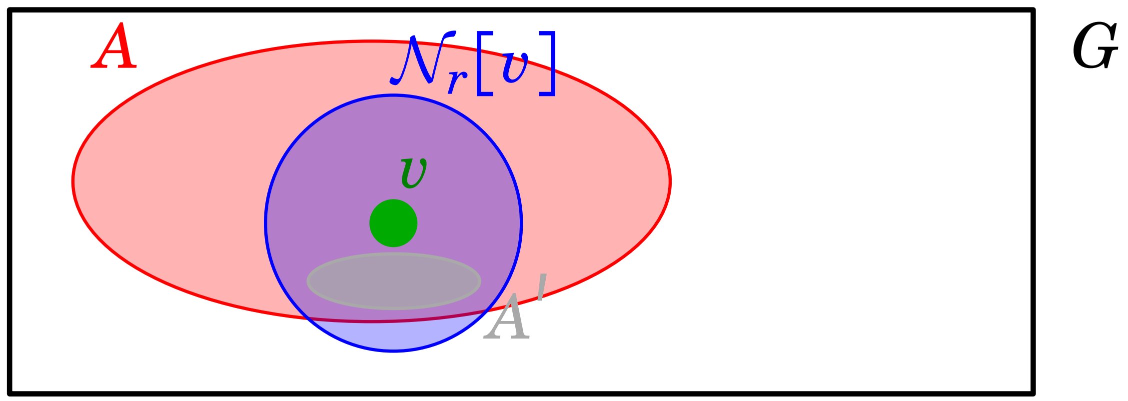

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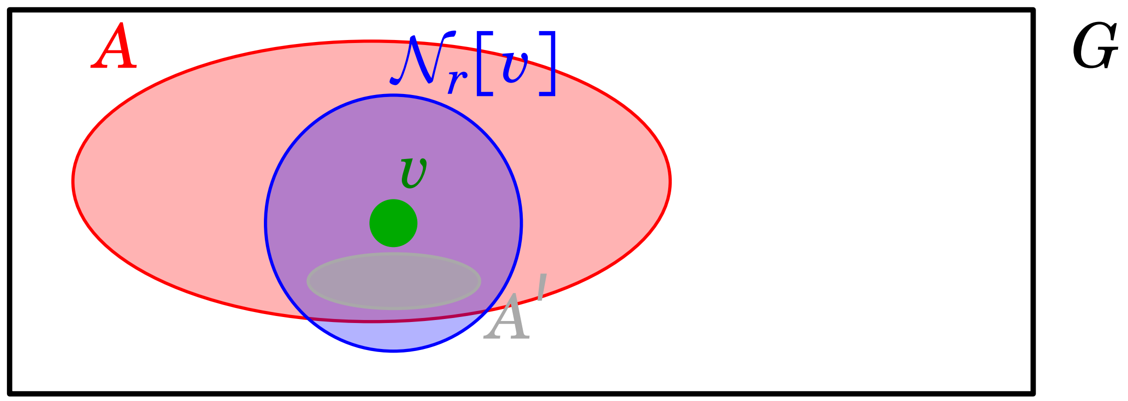

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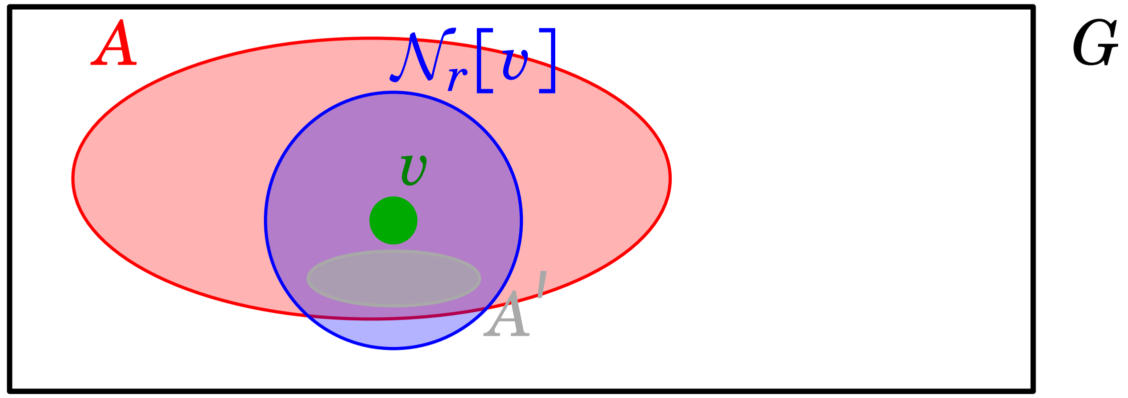

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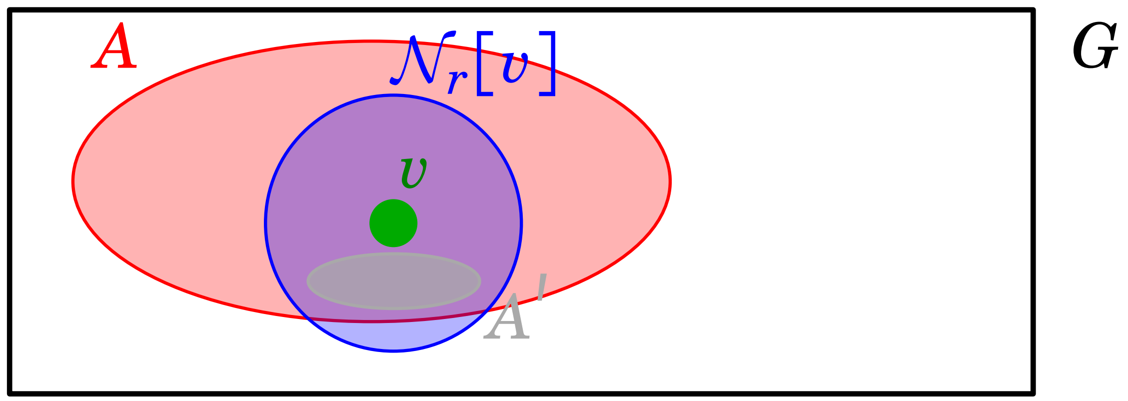

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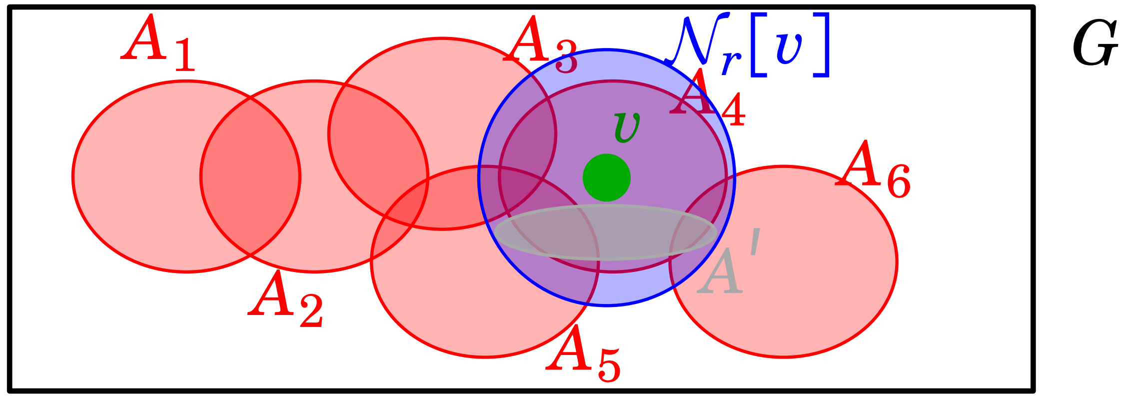
Property 2: on sequences of x-improving sets



- $A_1, \ldots, A_k \subseteq V$ sequence of x-improving sets
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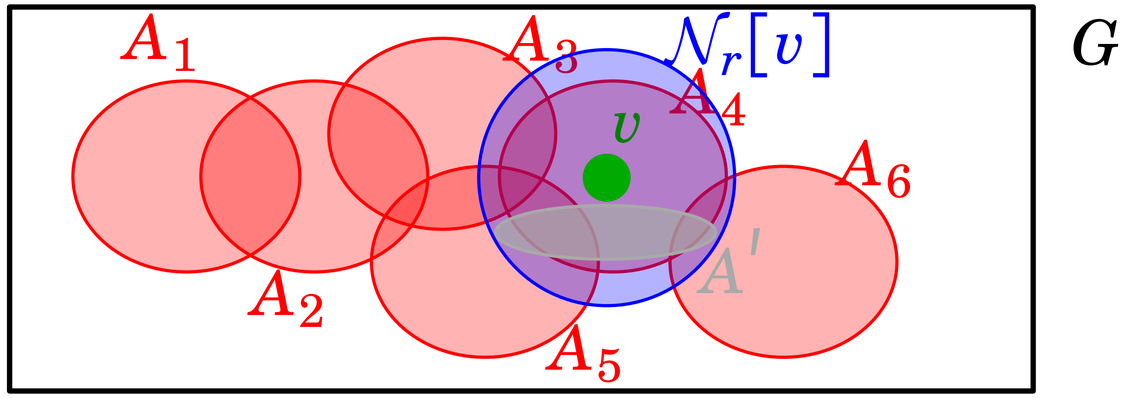

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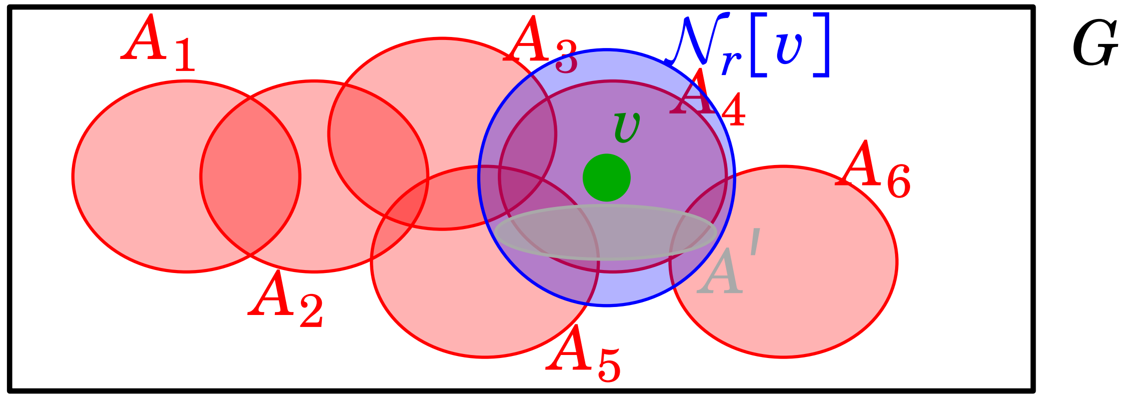

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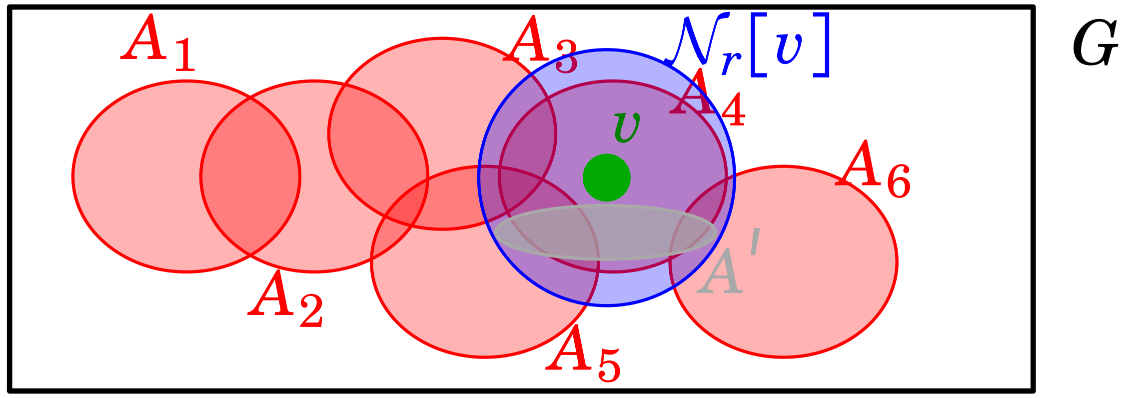

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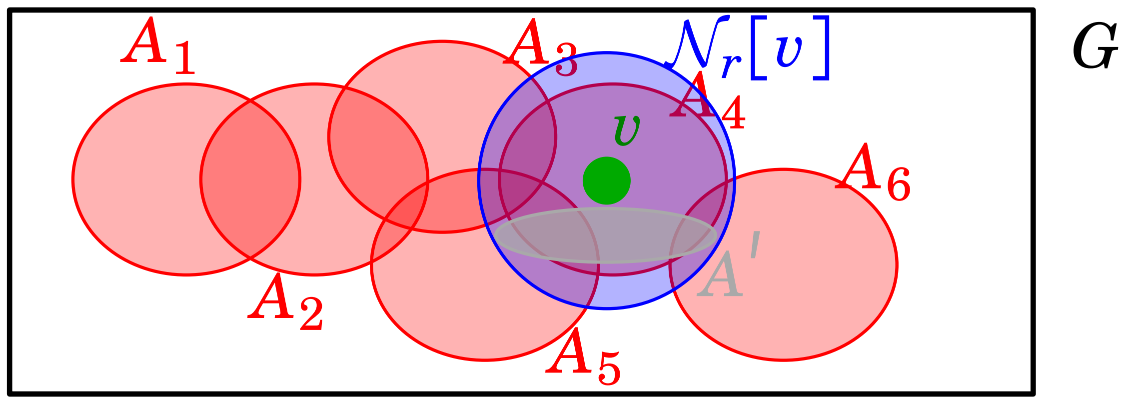

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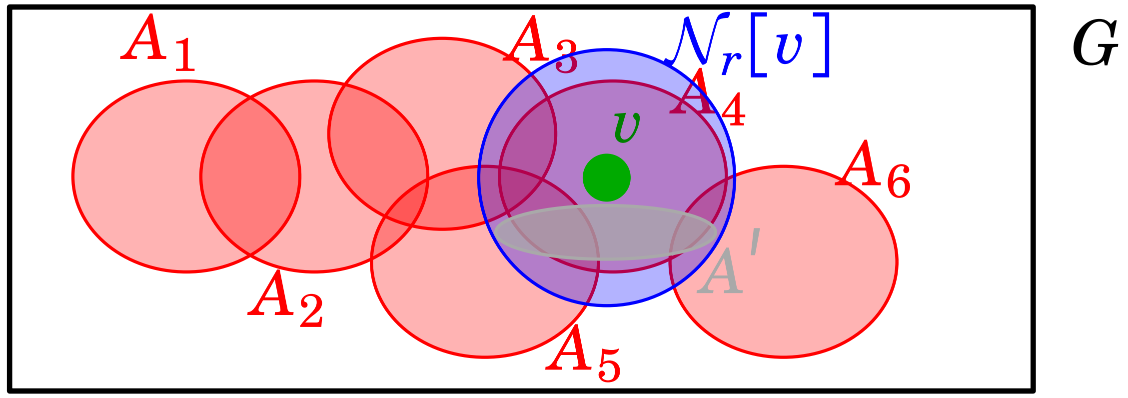

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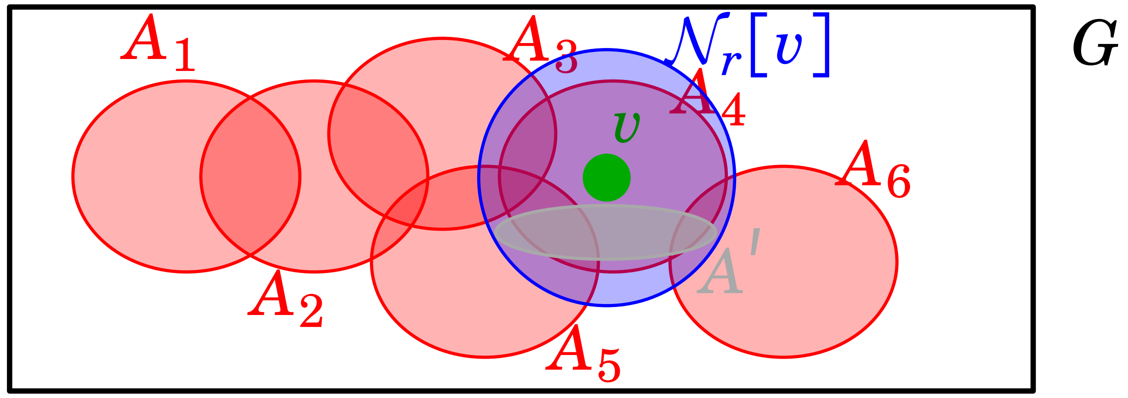

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- Similar to Lemma 1, but now to go back to G we need to multiply by $O(d)$ (diameter of the A_is)

Proof of Claim 2

- $\lambda_1 = 1/4$, $\varepsilon = \lambda/(2000 \log n)$, $\alpha = \Theta(\varepsilon^2/\log^2 n)$

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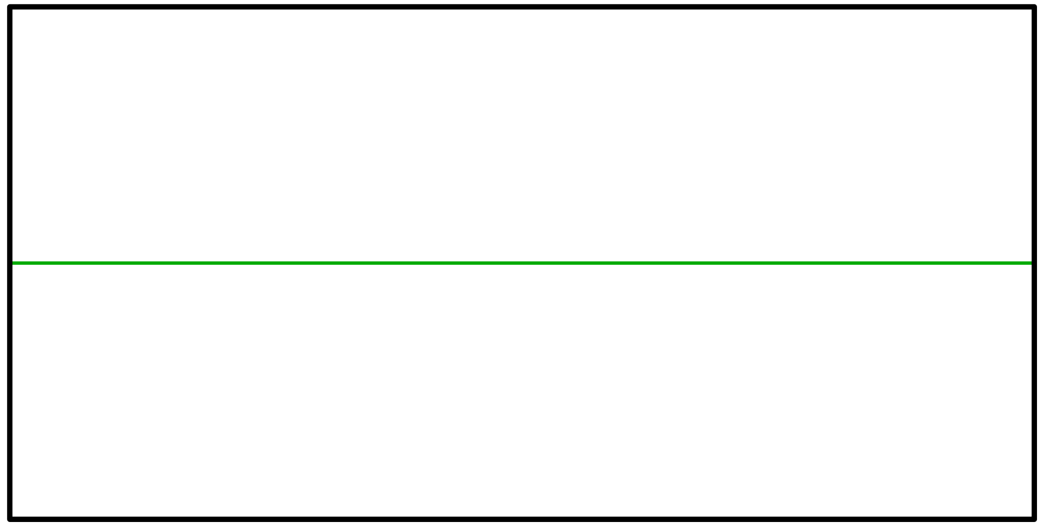

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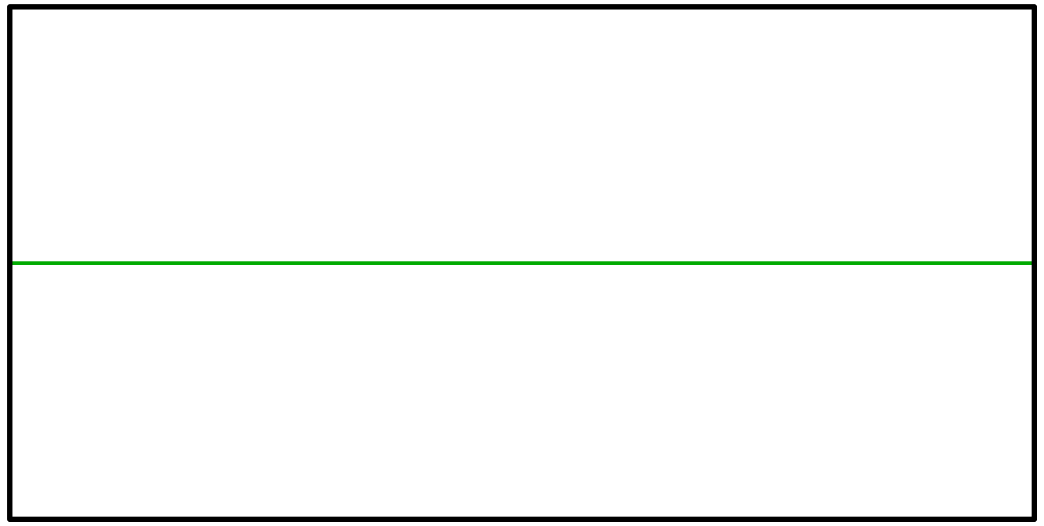

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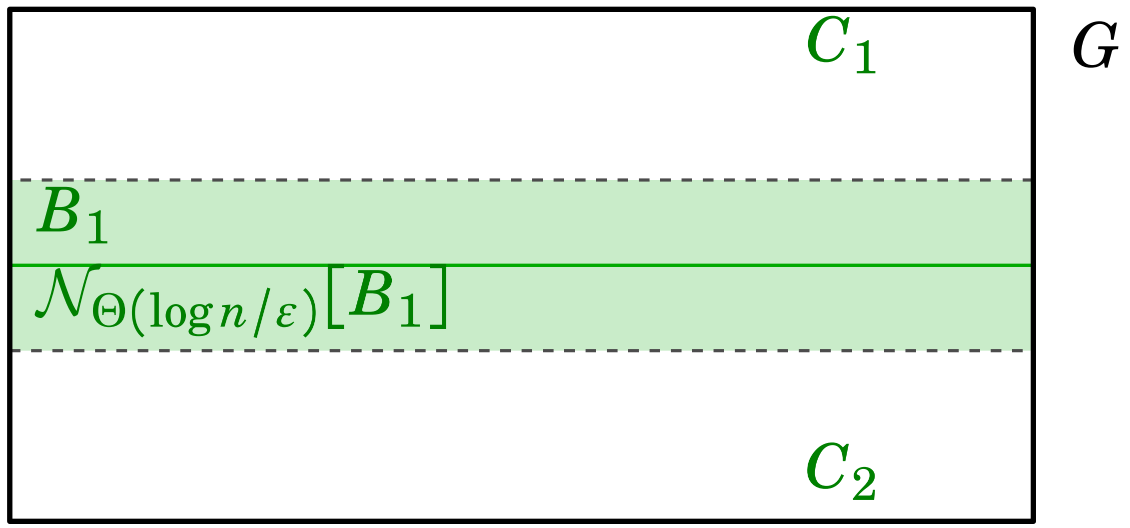

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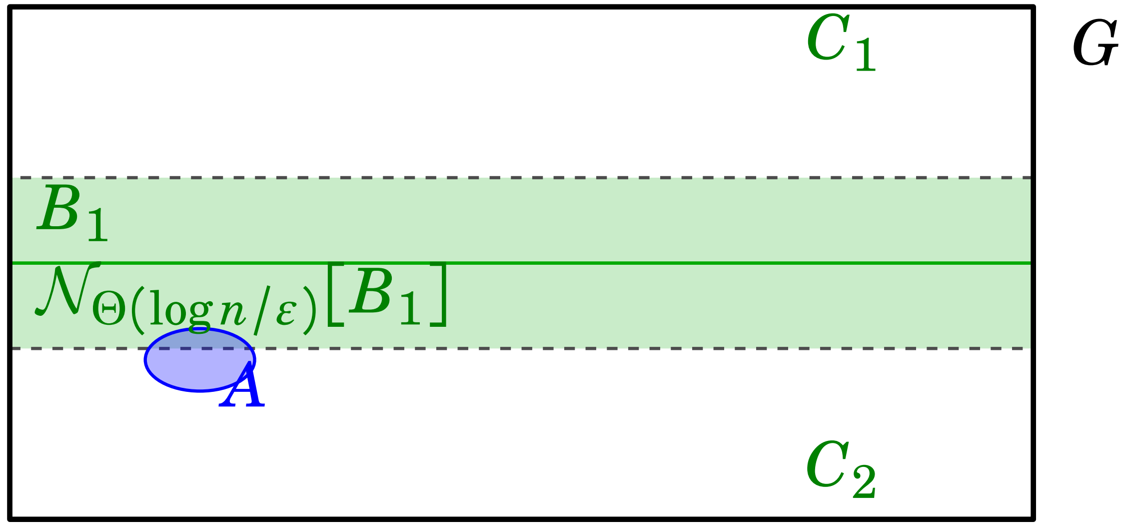

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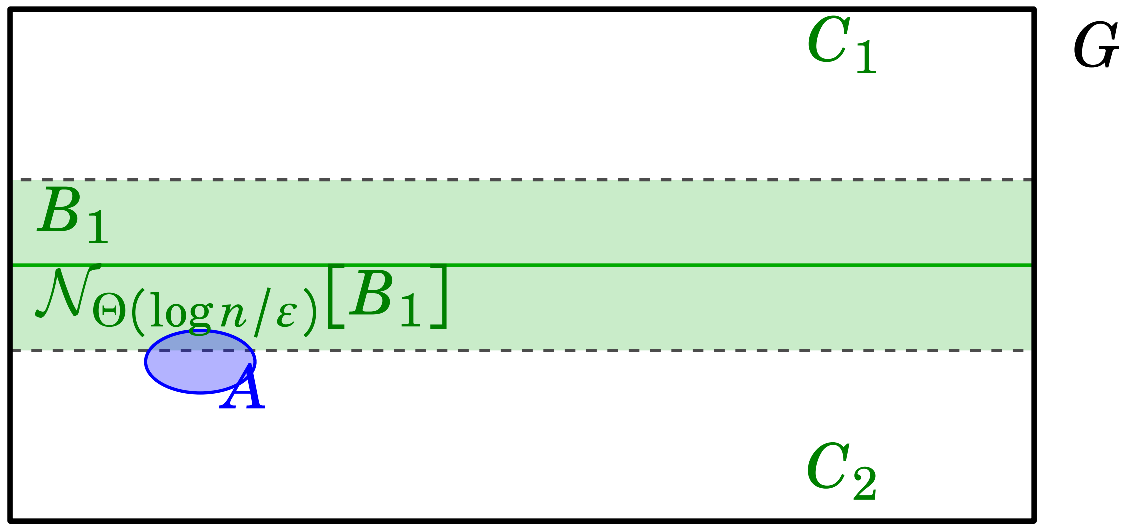

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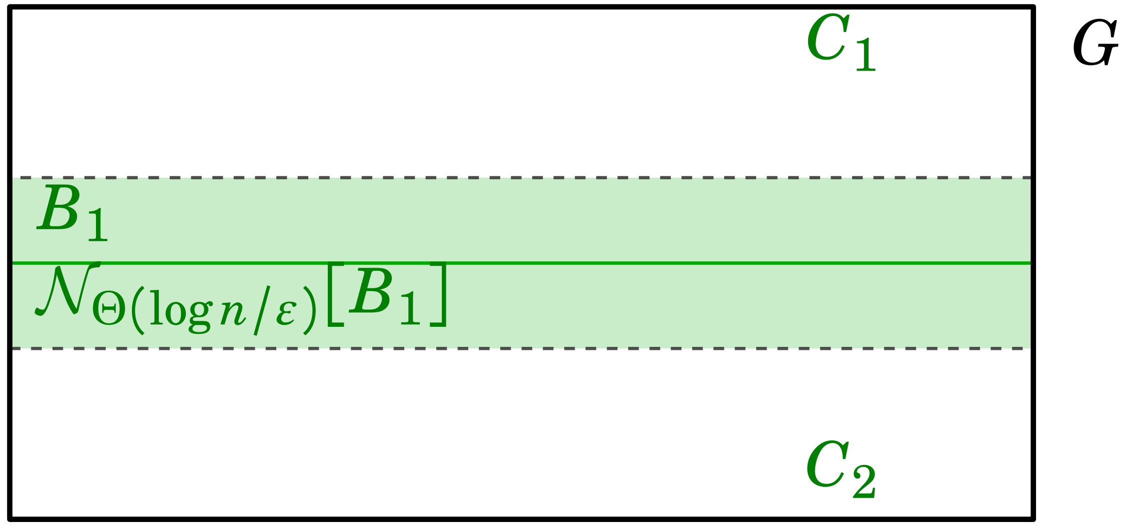

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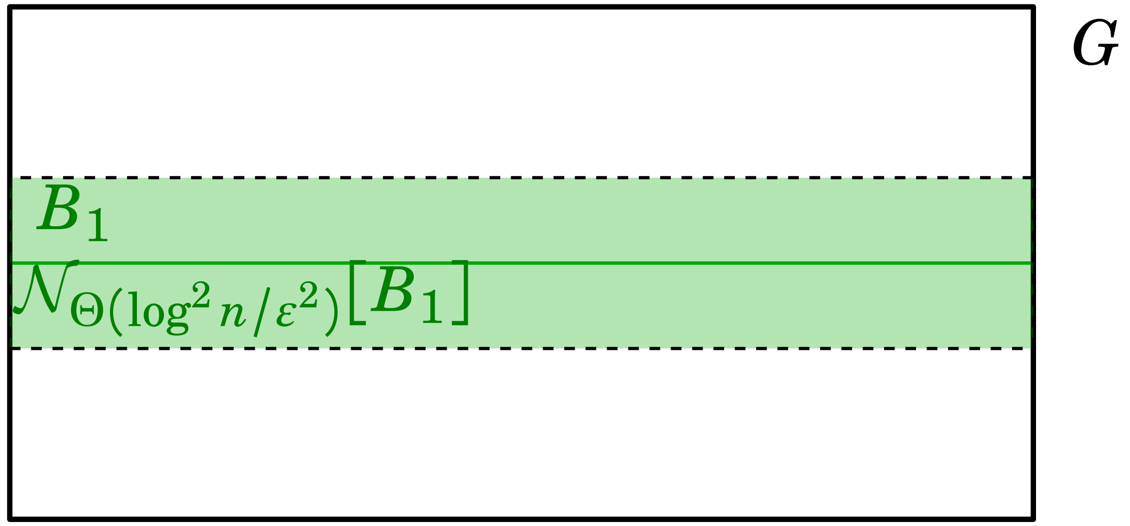

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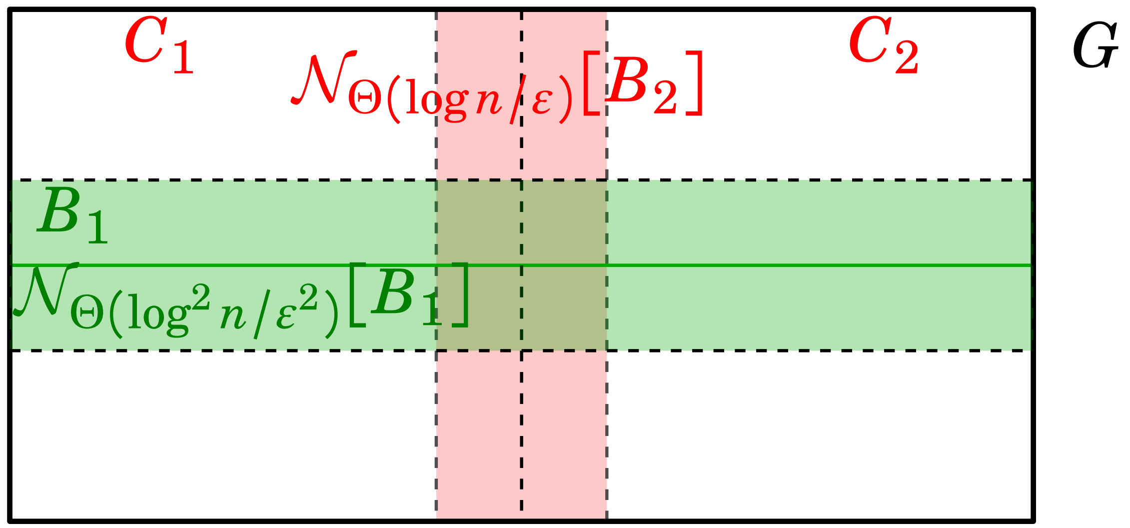

Proof of Claim 2

- $\lambda_1 = 1/4$, $\varepsilon = \lambda/(2000\log n)$, $\alpha = \Theta(\varepsilon^2/\log^2 n)$

Claim 2: *After phase i, any MIS with IR $\geq \lambda_i$ of diameter $O(\log n/\varepsilon)$ lies in $\mathcal{N}_{\Theta(\log n/\varepsilon)}[B_i] \cap [\cap_{j\leq i-1}\mathcal{N}_{\Theta(\log^2 n/\varepsilon^2)}[B_j]]$*

Proof 2: By induction on the phases.

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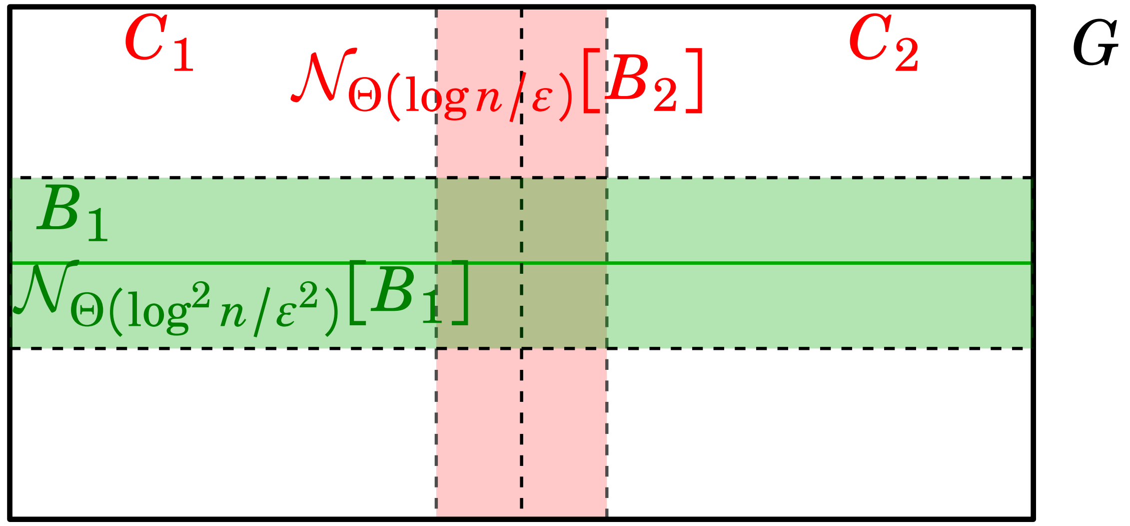

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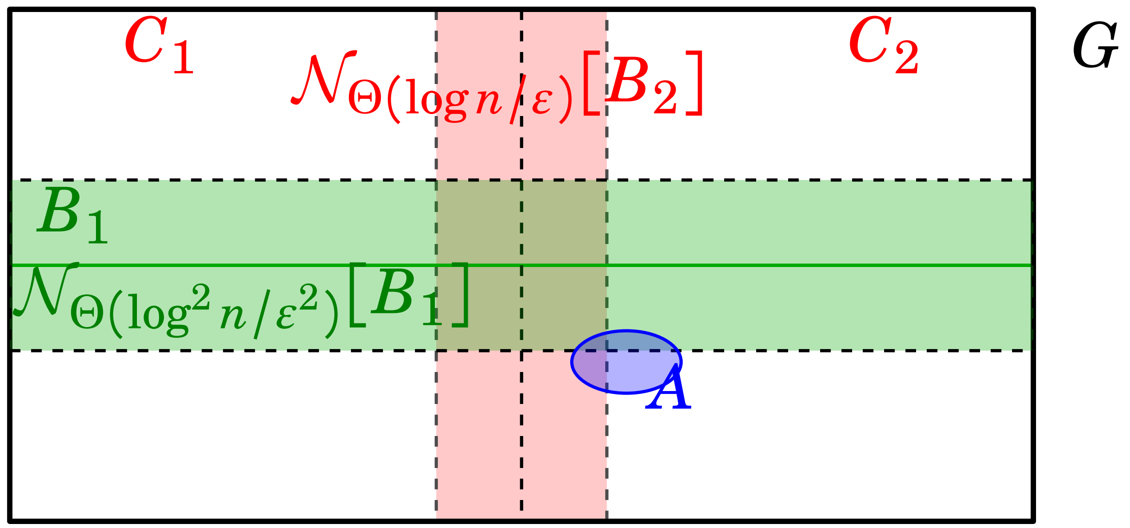

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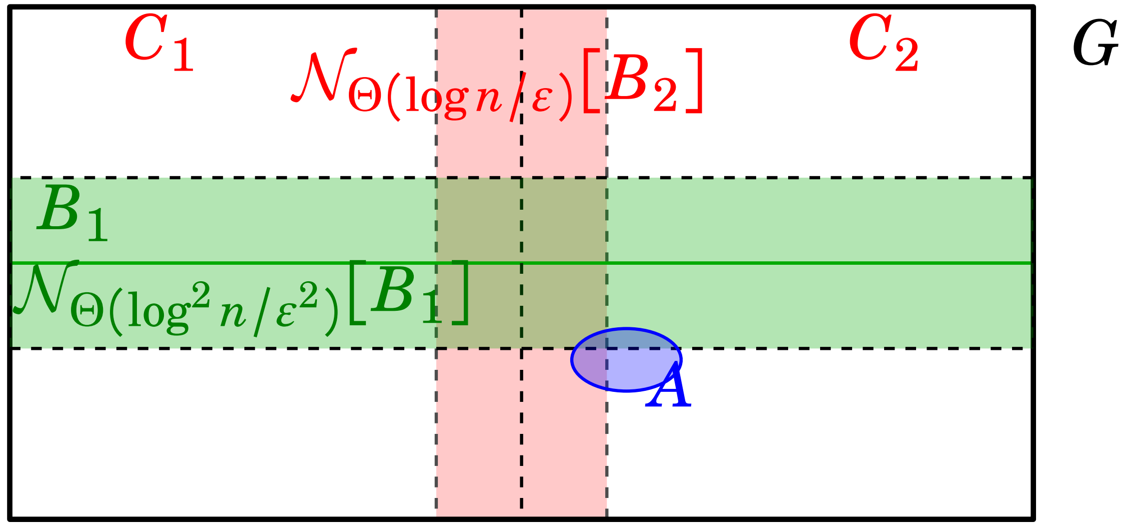

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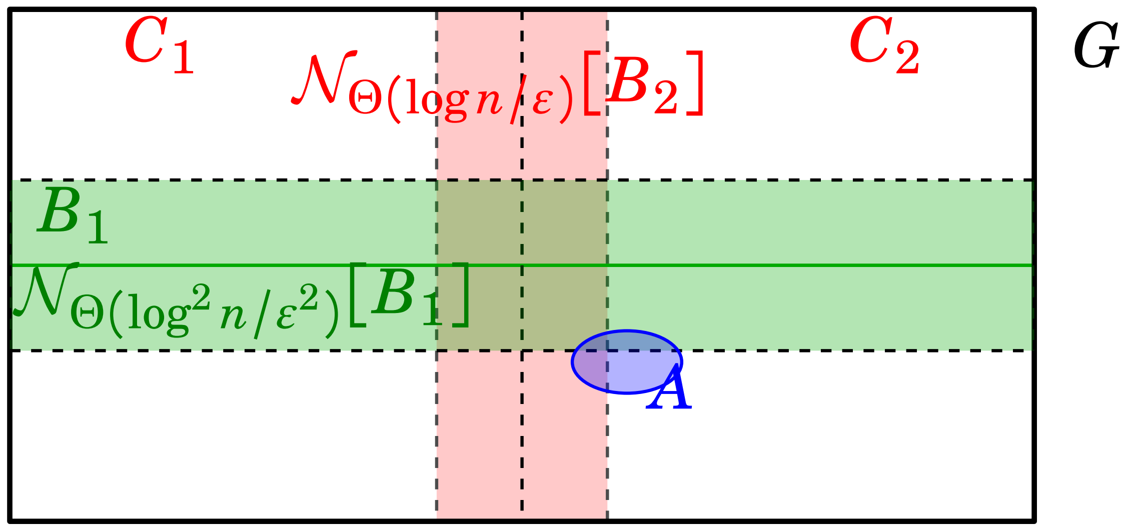

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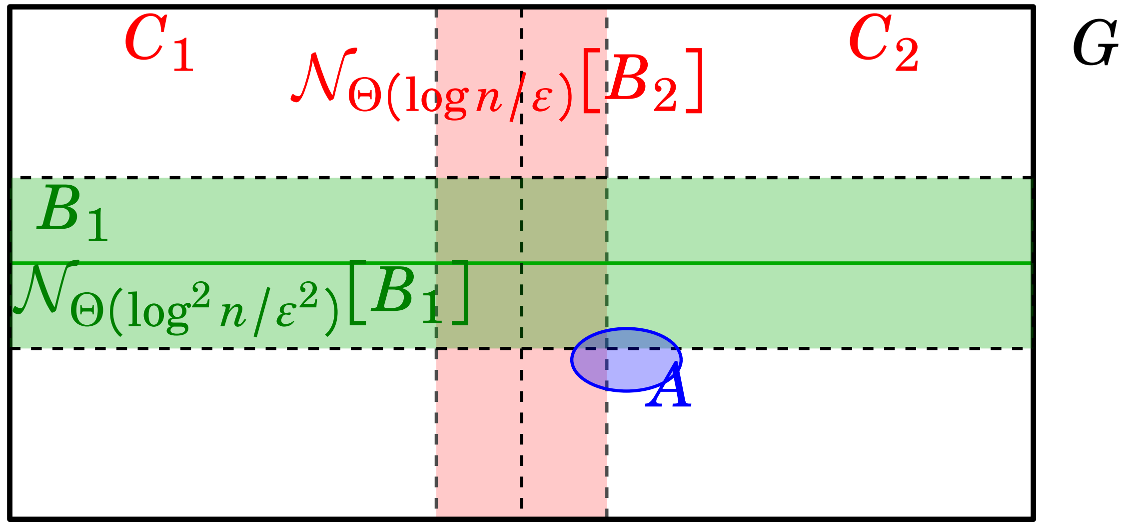

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- $\lambda_1 = 1/4$, $\varepsilon = \lambda/(2000\log n)$, $\alpha = \Theta(\varepsilon^2/\log^2 n)$

Claim 2: *After phase i, any MIS with IR ≥ λ_i of diameter $O(\log n/\varepsilon)$ lies in* $\mathcal{N}_{\Theta(\log n/\varepsilon)}[B_i] \cap [\cap_{j\le i-1}\mathcal{N}_{\Theta(\log^2 n/\varepsilon^2)}[B_j]]$

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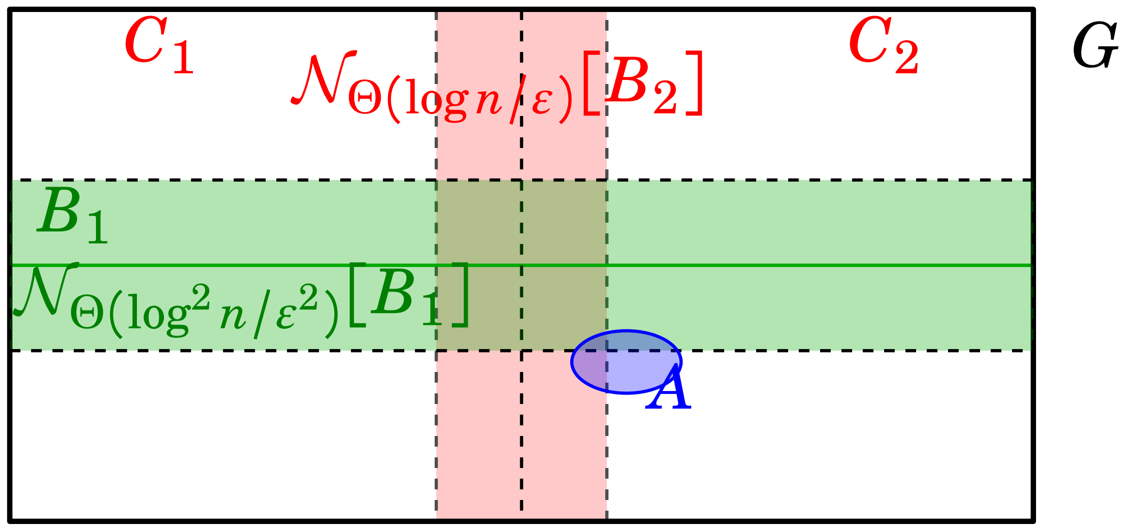

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- Property 2: $\exists$ min. imp. set A' inside $\mathcal{N}_{O(\log^2 n/\varepsilon^2)}[A]$ with IR$(A') \ge \lambda_{j+1} - \varepsilon \ge \lambda_j$

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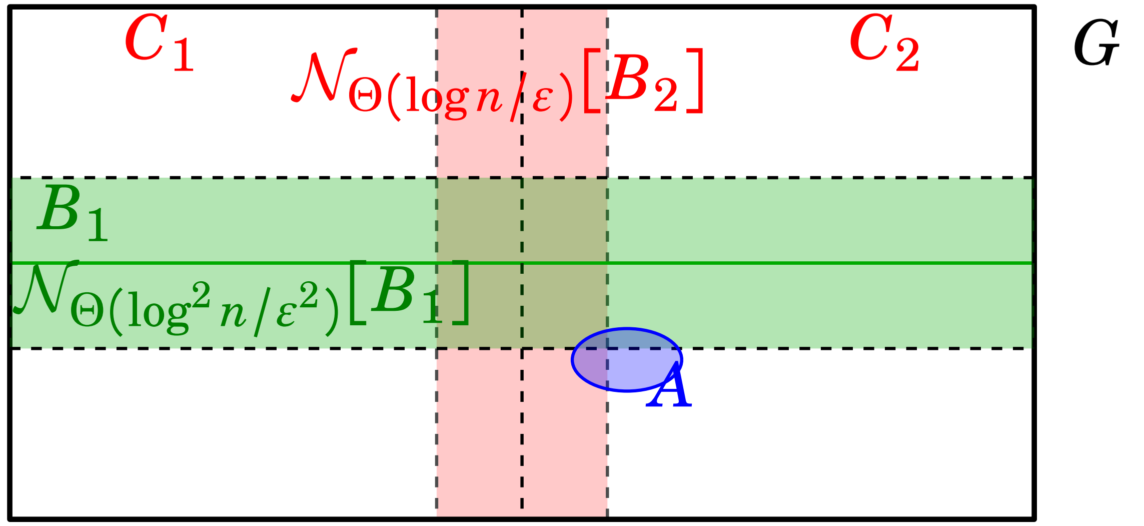

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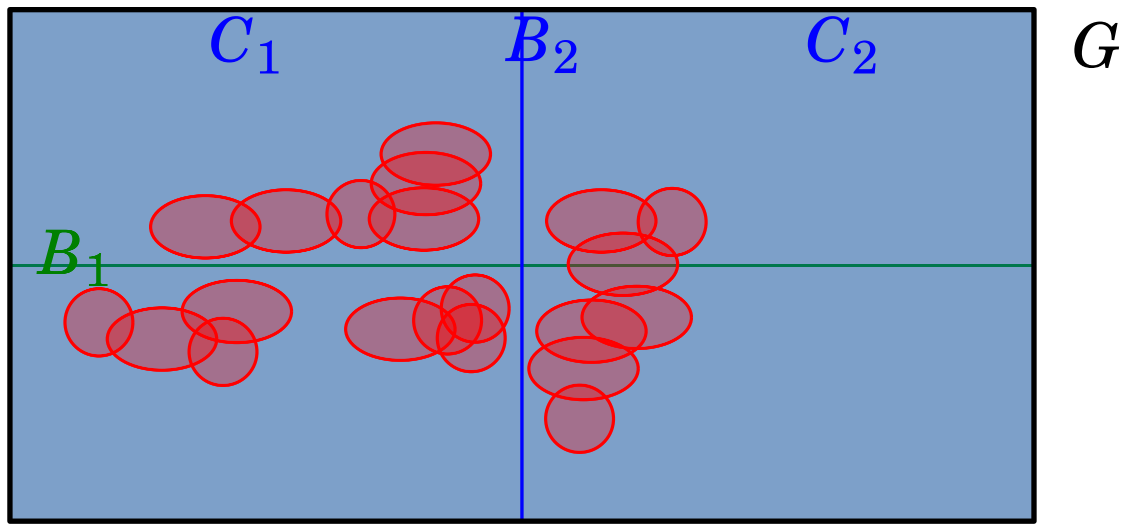
- $\lambda_1 = 1/4$, $\varepsilon = \lambda/(2000 \log n)$, $\alpha = \Theta(\varepsilon^2/\log^2 n)$



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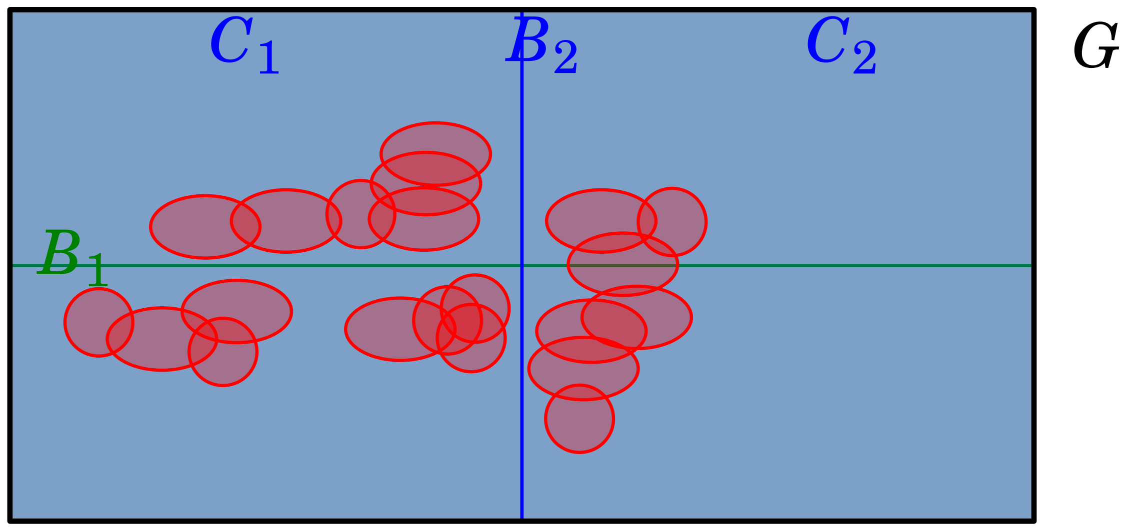
From $\mathcal{MPX}$: *For each* $v \in V$, *with probability* $\geq 1/2$ *there exists a cluster* C *such that* $\mathcal{N}_{\Theta(1/\alpha)}[v] \subseteq C$.



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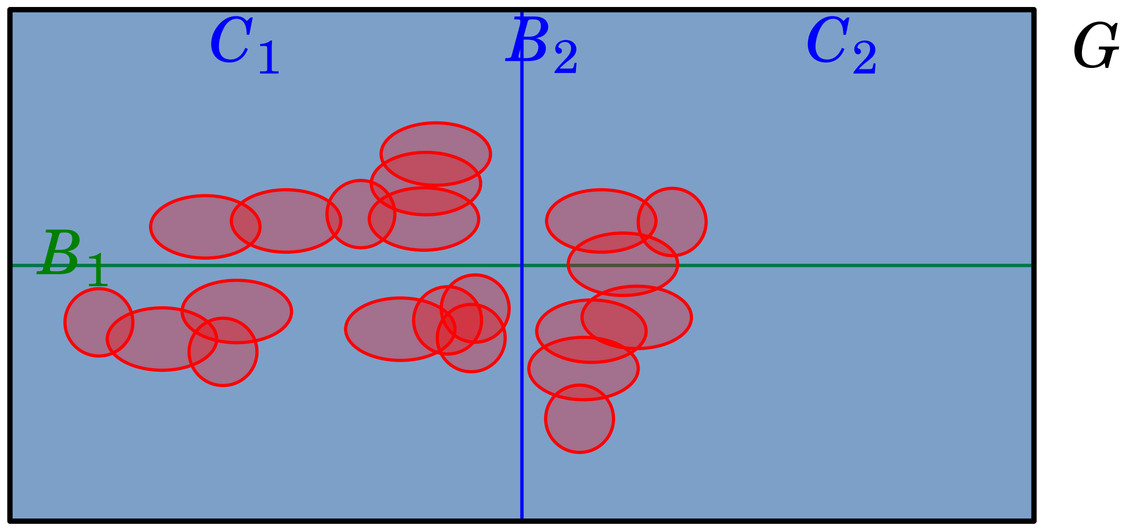

Proof of Claim 3

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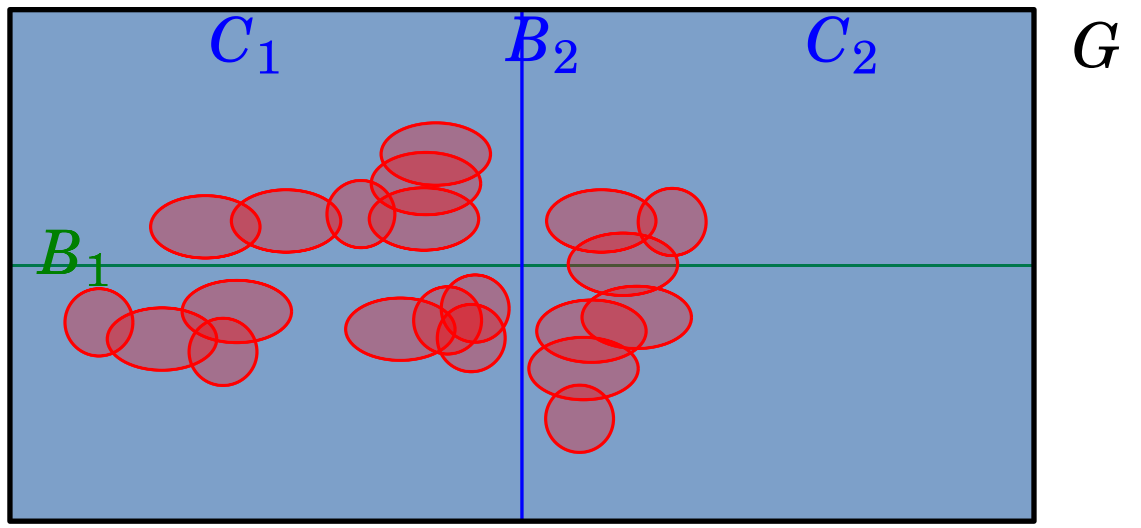
Claim 3: *With probability* $\geq 1 - 1/n^{10}$, *there is no error after* $100\log n$ *phases*



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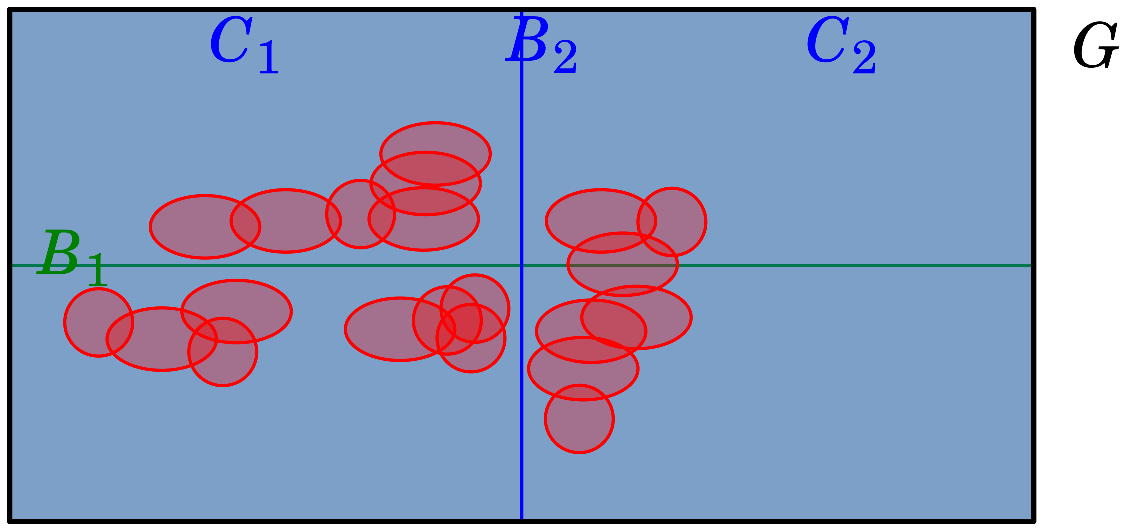
Claim 3: *With probability* $\geq 1 - 1/n^{10}$, *there is no error after* $100 \log n$ *phases*

Proof 3: By contradiction, there is an error at the end of phase $100 \log n$. Note that $\lambda_{100 \log n} \leq 3/4$.



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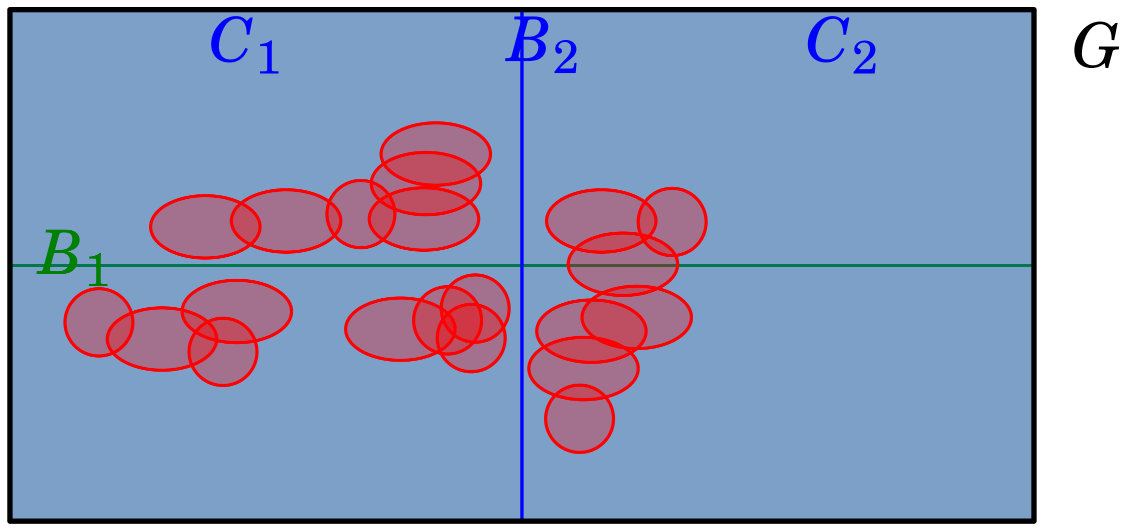
- Node v colored with green with $> \deg(v)/2$ neighbors of color green



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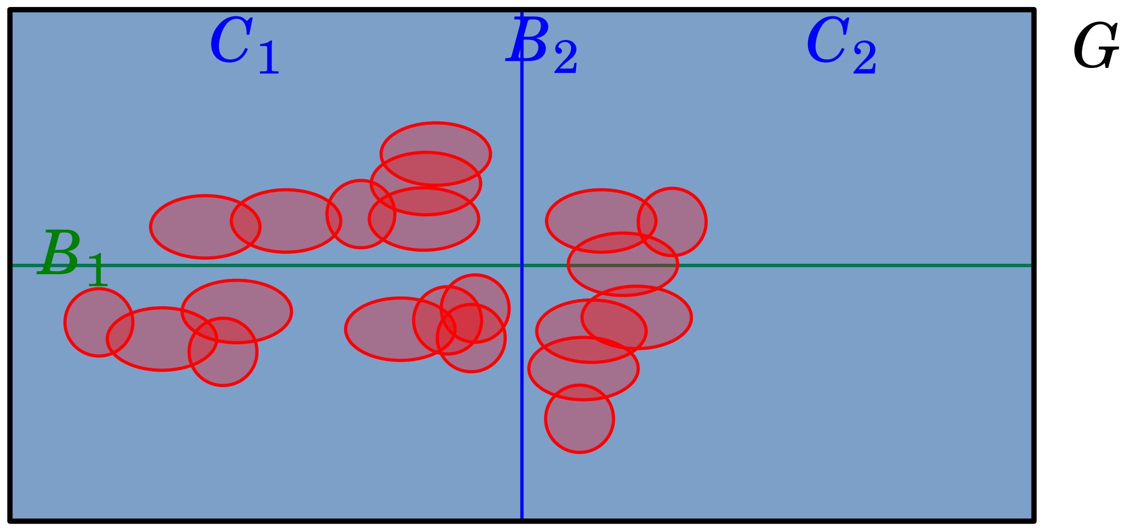

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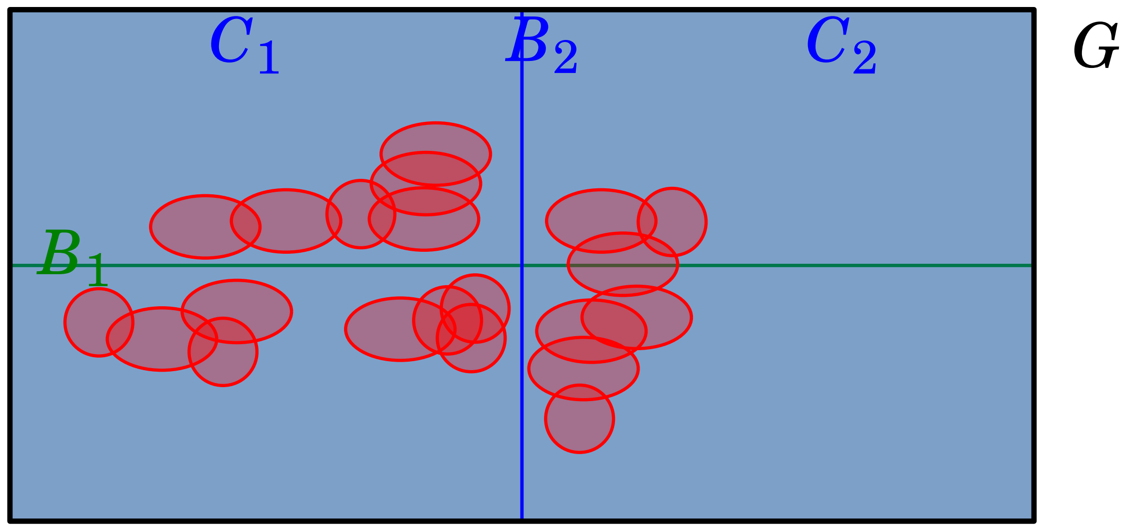
- $\{v\}$ is an improving set, and IR$(\{v\}) \geq 1$
- By $\mathcal{MPX}$ guarantees, $\exists i \leq 100 \log n$ such that $\mathcal{N}_{\Theta(1/\alpha)}[v]$ is contained in some cluster in phase i w.p. $1 - 1/2^{100 \log n - 1} \geq 1 - 1/n^{99}$



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- $\alpha = \Theta(\varepsilon^2/\log^2 n)$ is chosen large enough so that Claim 2 is contradicted