Invariant tensors and asymptotic rank of small tensors

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Outline

- Based on earlier and ongoing work with **Mateusz Michałek** (Konstanz)
- Talk outline:
  1. Tensors, tensor rank, asymptotic rank/tensor exponents
  2. Relevance of tensor exponents in the study of arithmetic circuits and algorithms
  3. The asymptotic rank conjecture
  4. Invariant tensors
  5. A candidate approach for rank upper bounds via orbits of subgroups (work in progress)

Preliminaries: Tensors

- We work in coordinates, all tensors have order three
- An element $T \in \mathbb{C}^{d\times d\times d} = \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ is a **tensor** of **shape** $d \times d \times d$
- For $i, j, k \in [d]$, we write $T_{i,j,k}$ for the **entry** of T at **position** (i, j, k)
- *Example.*
  The $4 \times 4 \times 4$ tensor MM_2 is displayed below:

$$\mathrm{MM}_2 = \left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1 \end{array}\right]$$

Preliminaries: Tensor rank

- A tensor $T \in \mathbb{C}^{d\times d\times d}$ has **rank one** if there exist three nonzero vectors $a, b, c \in \mathbb{C}^d$ such that $T = a \otimes b \otimes c$; or, what is the same, $T_{i,j,k} = a_i b_j c_k$ for all $i, j, k \in [d]$
- The **rank** $\mathrm{R}(T)$ of a tensor $T \in \mathbb{C}^{d\times d\times d}$ is the least nonnegative integer r such that T can be written as a sum of r rank one tensors
- We have $0 \leq \mathrm{R}(T) \leq d^2$; it is NP-hard to compute $\mathrm{R}(T)$ for given T (Håstad 1990)
- *Example.* The rank of MM_2 is 7 (Strassen 1969)

Preliminaries: Kronecker product and Kronecker powers

- Let $S \in \mathbb{C}^{d\times d\times d}$ and $T \in \mathbb{C}^{e\times e\times e}$ be tensors
- The **Kronecker product** $S \otimes T \in \mathbb{C}^{de\times de\times de}$ is defined for all $i, j, k \in [d]$ and $u, v, w \in [e]$ by

$$(S \otimes T)_{ie+u,\, je+v,\, ke+w} = S_{i,j,k} T_{u,v,w}$$

- For $S \in \mathbb{C}^{d\times d\times d}$ and a positive integer n, we write $S^{\otimes n} \in \mathbb{C}^{d^n\times d^n\times d^n}$ for the Kronecker product of n copies of S
- We say that $S^{\otimes n}$ is the n^{th} **Kronecker power** of S

Example: Kronecker powers

$$S = \left[\begin{array}{c|c} 0\,1 & 1\,0 \\ 1\,0 & 0\,0 \end{array}\right]$$

$$S^{\otimes 2} = \left[\begin{array}{c|c|c|c} 0\,0\,0\,1 & 0\,0\,1\,0 & 0\,1\,0\,0 & 1\,0\,0\,0 \\ 0\,0\,1\,0 & 0\,0\,0\,0 & 1\,0\,0\,0 & 0\,0\,0\,0 \\ 0\,1\,0\,0 & 1\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 \\ 1\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 \end{array}\right]$$

$$S^{\otimes 3} = \left[\begin{array}{c|c|c|c|c|c|c|c}
00000001 & 00000010 & 00000100 & 00001000 & 00010000 & 00100000 & 01000000 & 10000000 \\
00000010 & 00000000 & 00001000 & 00000000 & 00100000 & 00000000 & 10000000 & 00000000 \\
00000100 & 00001000 & 00000000 & 00000000 & 01000000 & 10000000 & 00000000 & 00000000 \\
00001000 & 00000000 & 00000000 & 00000000 & 10000000 & 00000000 & 00000000 & 00000000 \\
00010000 & 00100000 & 01000000 & 10000000 & 00000000 & 00000000 & 00000000 & 00000000 \\
00100000 & 00000000 & 10000000 & 00000000 & 00000000 & 00000000 & 00000000 & 00000000 \\
01000000 & 10000000 & 00000000 & 00000000 & 00000000 & 00000000 & 00000000 & 00000000 \\
10000000 & 00000000 & 00000000 & 00000000 & 00000000 & 00000000 & 00000000 & 00000000
\end{array}\right]$$

Tensor exponents and asymptotic rank

- The **exponent** $\sigma(T)$ of a tensor $T \in \mathbb{C}^{d\times d\times d}$ is the infimum of all $\sigma > 0$ such that $\mathrm{R}(T^{\otimes n}) \leq d^{\sigma n + o(n)}$ holds
- Equivalently, the **asymptotic rank** of T is $\tilde{\mathrm{R}}(T) = \lim_{n\to\infty} \mathrm{R}(T^{\otimes n})^{1/n} = d^{\sigma(T)}$
- Exponents of *constant-size* tensors are fundamental to the study of algorithms
  - The exponent ω of square matrix multiplication satisfies $\omega = 2\sigma(\mathrm{MM}_2)$ [Strassen 1986/1988]
  - The set cover conjecture fails if the exponent of a specific $7 \times 7 \times 7$ tensor is sufficiently close to 1 [Björklund & K. 2024] (see also [Pratt 2024])
  - If specific large but constant-size tensors have their exponents sufficiently close to 1, then the chromatic number of a given n-vertex graph can be computed in $O(1.99982^n)$ time [Björklund, Curticapean, Husfeldt, K., & Pratt 2025]
  - If specific large but constant-size tensors have their exponents sufficiently close to 1, then the permanent of an $n \times n$ matrix can be computed with a uniform arithmetic circuit of size $O(1.9^n)$ [Björklund, K., Koana, & Nederlof 2025]

Example: The $7 \times 7 \times 7$ tensor [Björklund & K. 2024]

$$
Q_7 = \left[\begin{array}{c|c|c|c|c|c|c}
0000000 & 0000001 & 0000010 & 0000100 & 0001000 & 0010000 & 0100000 \\
0000001 & 0000000 & 0000100 & 0000000 & 0010000 & 0000000 & 1000000 \\
0000010 & 0000100 & 0000000 & 0000000 & 0100000 & 1000000 & 0000000 \\
0000100 & 0000000 & 0000000 & 0000000 & 1000000 & 0000000 & 0000000 \\
0001000 & 0010000 & 0100000 & 1000000 & 0000000 & 0000000 & 0000000 \\
0010000 & 0000000 & 1000000 & 0000000 & 0000000 & 0000000 & 0000000 \\
0100000 & 1000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000
\end{array}\right]
$$

If $\sigma(Q_7) \leq 1.001$ then the set cover conjecture is false

[We know that $\sigma(Q_7) \leq 1.069$]

The asymptotic rank conjecture

- Define the **worst-case** tensor exponent for $d \times d \times d$ tensors by

$$\sigma(d) = \sup_{T \in \mathbb{C}^{d \times d \times d}} \sigma(T)$$

- It is immediate that $\sigma(1) = 1$; it is a nontrivial consequence of the geometry of tensors that $\sigma(2) = 1$; already $\sigma(3)$ is unknown—it is known that $\sigma(3) = 1$ implies $\omega = 2$
- Strassen (1988, implicit) has shown that $\sigma(d) \leq 2\omega/3$ for all $d \in \mathbb{Z}_{\geq 1}$; the following bold conjecture has been made by many
- **Conjecture. (Asymptotic rank conjecture)**
  For all $d \in \mathbb{Z}_{\geq 1}$ it holds that $\sigma(d) = 1$
- [Strassen (1994) has conjectured $\sigma(T) = 1$ for tight and concise tensors T.]

But how to approach the asymptotic rank conjecture?

Caveat. This talk does not give a proper survey—for background and recent work, cf. e.g. (Wigderson & Zuiddam 2023), (Christandl, Hoeberechts, Nieuwboer, Vrana, & Zuiddam 2025), (K. & Michałek 2025) as well as references therein

One possible approach: Invariant tensors

- Let $d, n \in \mathbb{Z}_{\geq 1}$
- Let us write S_n for the **symmetric group** on $[n]$
- We assume a permutation $g \in S_n$ **acts**
  1. on $[n]$ by permutation;
  2. on $[d]^n$ by permuting the entries of an n-tuple over $[d]$; and
  3. on $\mathbb{C}^{d^n \times d^n \times d^n}$ by permuting the rows, columns, and levels as in $[d]^n$
- A tensor $T \in \mathbb{C}^{d^n \times d^n \times d^n}$ is S_n-**invariant** if $gT = T$ for all $g \in S_n$
- Let us write $(\mathbb{C}^{d^n \times d^n \times d^n})^{S_n}$ for the set of all S_n-invariant tensors in $\mathbb{C}^{d^n \times d^n \times d^n}$
- **Theorem (K. & Michałek 2025).**
  For all $d \in \mathbb{Z}_{\geq 1}$ we have $\sigma(d) = \lim_{n\to\infty} \frac{1}{n} \log_d \max_{T \in (\mathbb{C}^{d^n \times d^n \times d^n})^{S_n}} R(T)$
- The space $(\mathbb{C}^{d^n \times d^n \times d^n})^{S_n}$ can be decomposed into smaller invariant subspaces using tools from representation theory—in this talk we will not enter into detailed discussion

A subapproach: Upper bounds via orbits of subgroups

- Focus on small d; e.g. $d = 2$ or $d = 3$ in particular
- While the vector space $\mathbb{C}^{d^n \times d^n \times d^n}$ has dimension d^{3n},
  the invariant subspace $(\mathbb{C}^{d^n \times d^n \times d^n})^{S_n}$ has dimension only $\binom{n+d^3-1}{d^3-1}$
- To show that tensors in the vector space $(\mathbb{C}^{d^n \times d^n \times d^n})^{S_n}$ have low rank,
  it suffices to present a basis such that each tensor T in the basis is the sum of
  short G-orbits of rank-one tensors for one or more permutation groups $G \leq S_n$
- The larger the group $G \leq S_n$, the shorter the orbits; a G-orbit has length at most $n!/|G|$
- A natural family of essentially maximal subgroups of S_n are the **Young subgroups**
  $S_\nu = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_p}$ for an integer partition $\nu = (n_1, n_2, \ldots, n_p)$ of n to p parts
- For $d = 2$, one can show that Young subgroups with at most two parts suffice to span
  the invariant space for any n, thus giving the already known $\sigma(2) = 1$
- For $d = 3$, work in progress ...

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