# The impact of curvature on the structure of geometric (intersection) graphs

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Thomas Bläsius, Jean-Pierre von der Heydt, Marcus Wilhelm, Geert van Wordragen

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29 August 2024

Aalto University
School of Science

# Overview

- Intro: Curvature, the hyperbolic plane,
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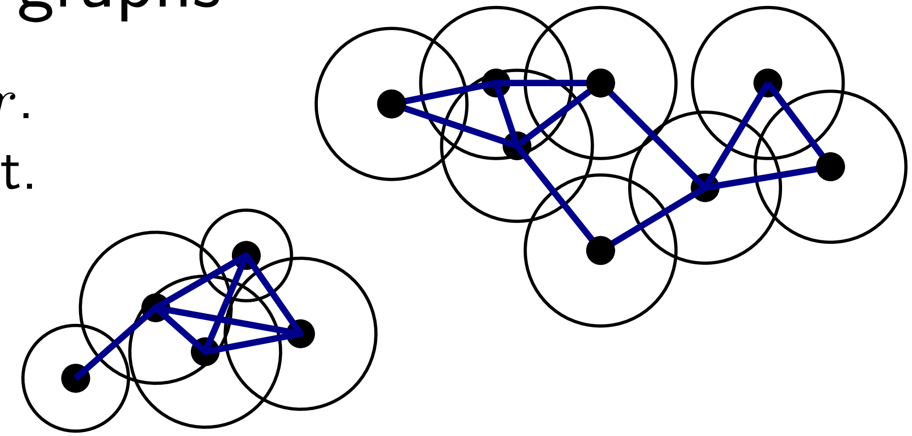
- Plan of attack: DP on noose hierarchy
- Voronoi, Delaunay
- Outerplanarity of Delaunay in $\mathbb{H}^2$
- (Musings on hyperbolic surfaces)

# Intersection graphs

$P \subset \mathbb{X}$, connect points if within distance $2r$.

$\Leftrightarrow$ connect if disks of radius $r$ intersect.



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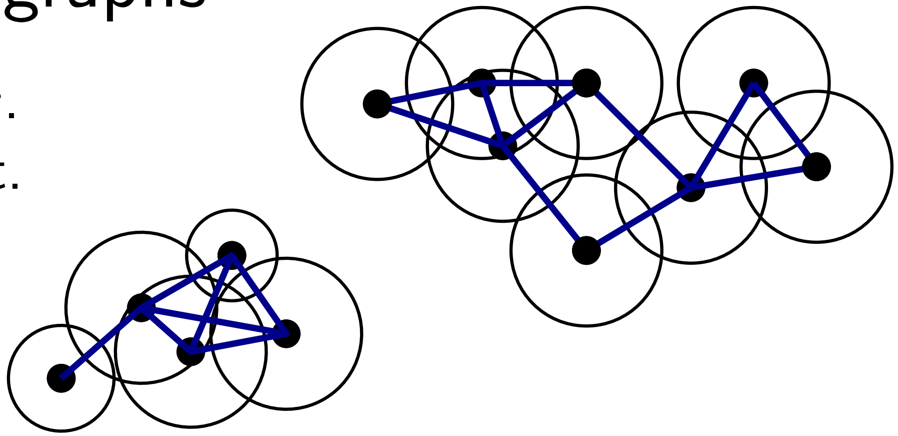

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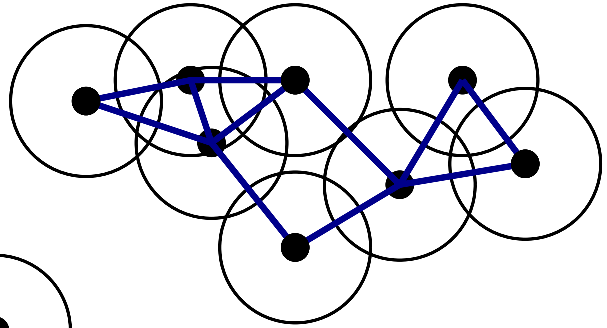

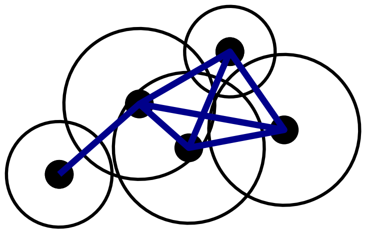

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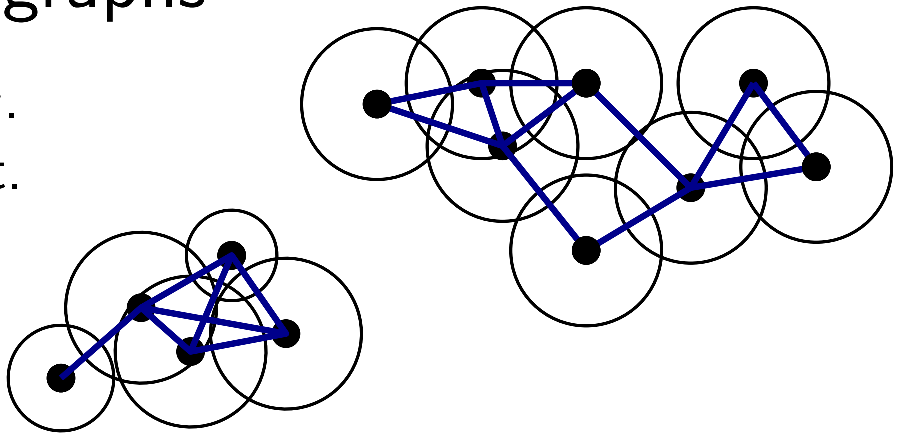

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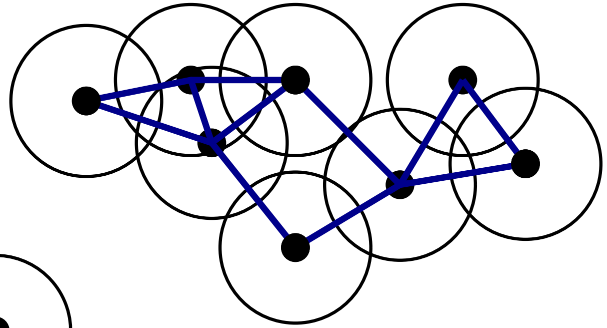

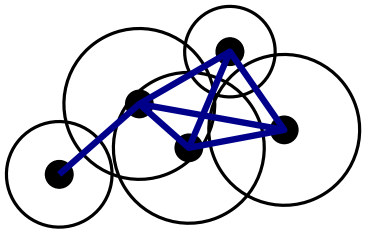

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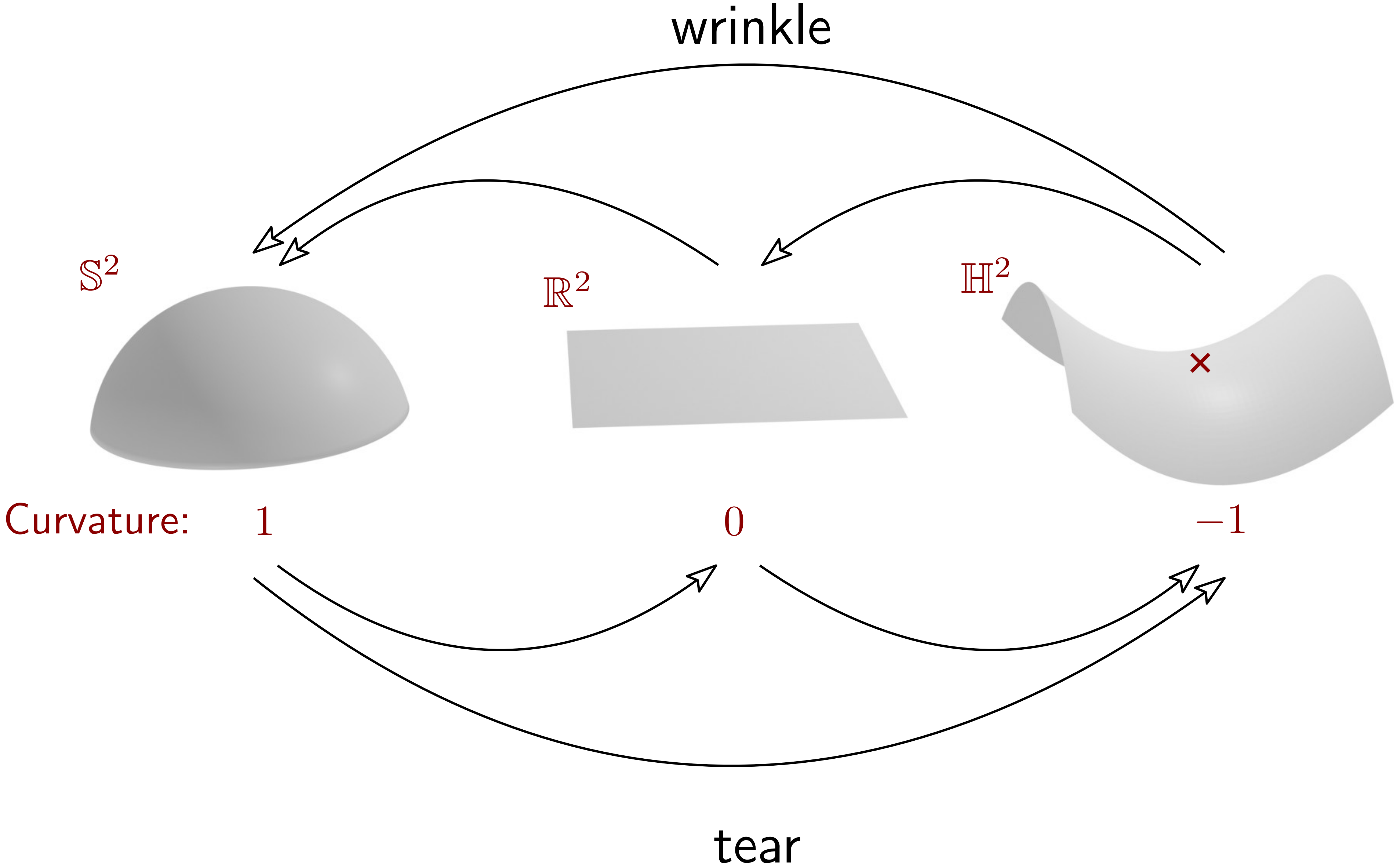
**Theorem (Chan)**
INDEPENDENT SET in DG $(1-\varepsilon)$-approxiamted in $n^{O(1/\varepsilon)}$ time in $\mathbb{R}^2$.

Both are conditionally optimal, even for $r \equiv 1$.

# Curvature: surfaces made of wood or paper



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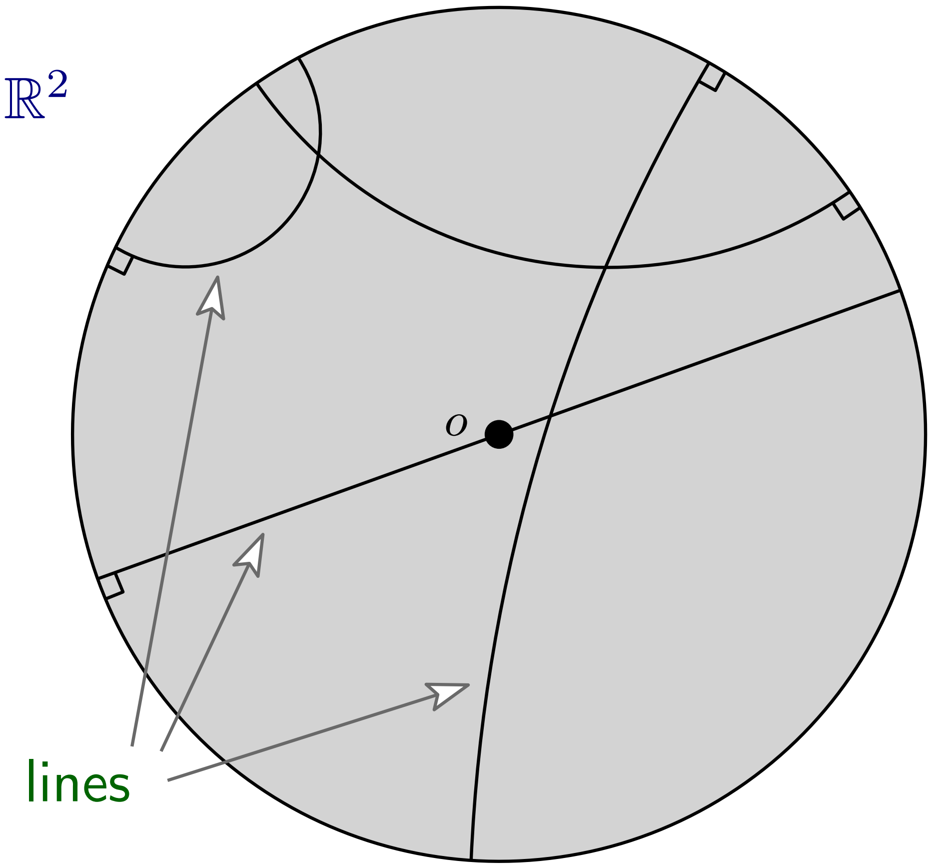
$$\text{dist}(u,v) = \cosh^{-1}\left(1 + 2\frac{\|u-v\|^2}{(1-\|u\|^2)(1-\|v\|^2)}\right)$$

Poincaré disk model: open unit disk in $\mathbb{R}^2$



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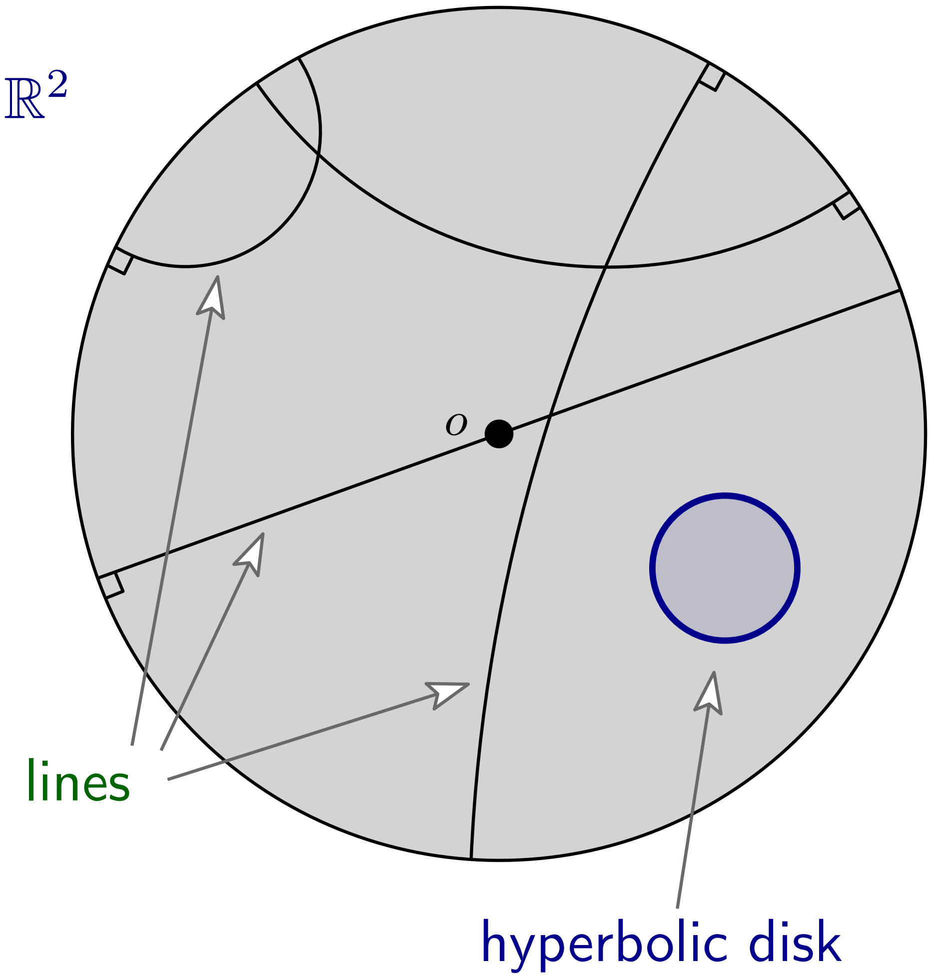
- disk of (hyperb.) radius $r > 1$
has (hyperb.) area $\Theta(e^r)$



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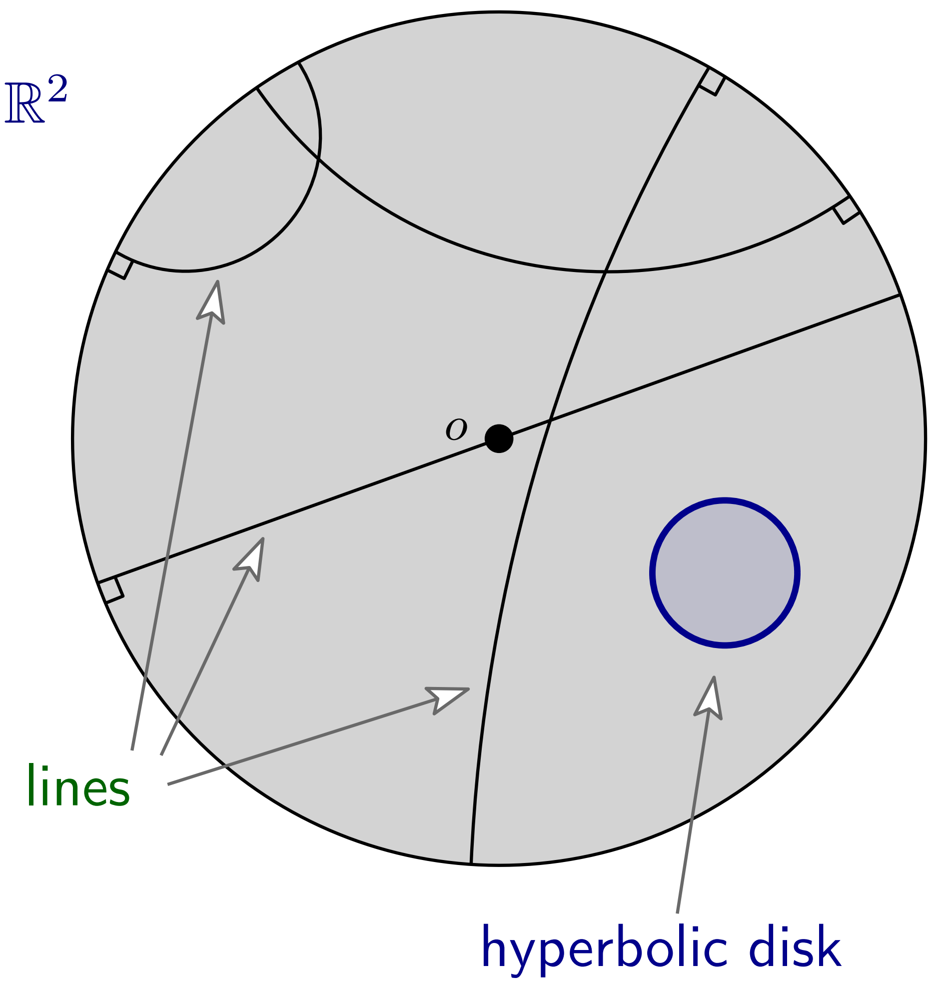

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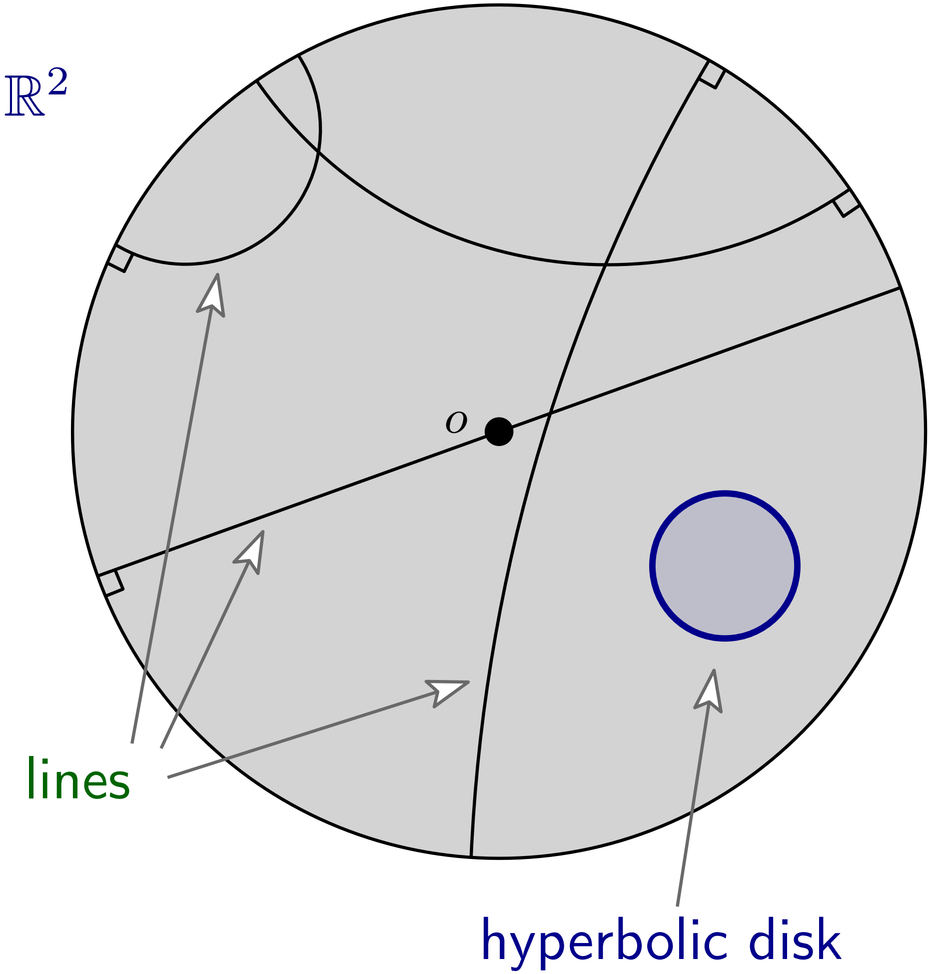

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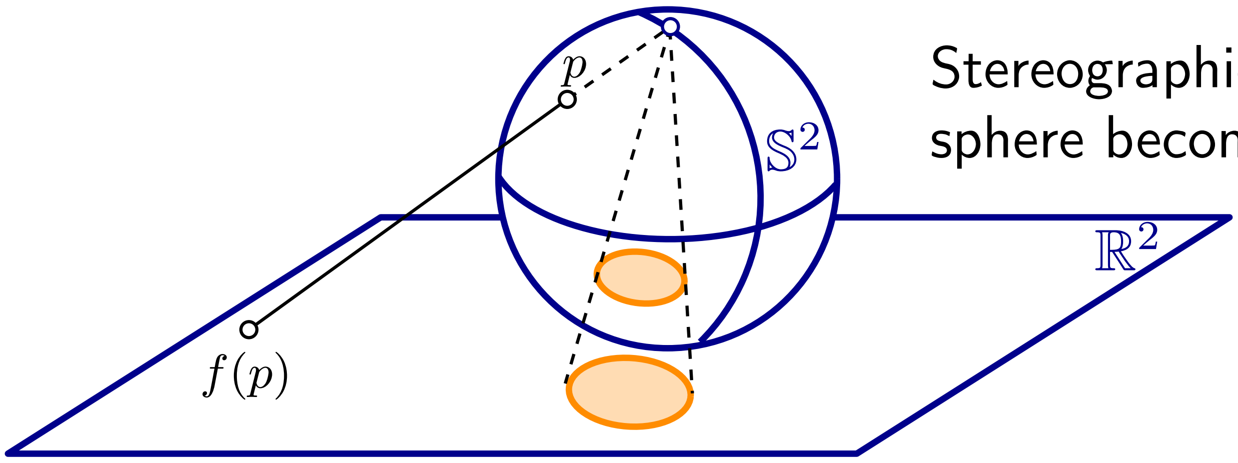


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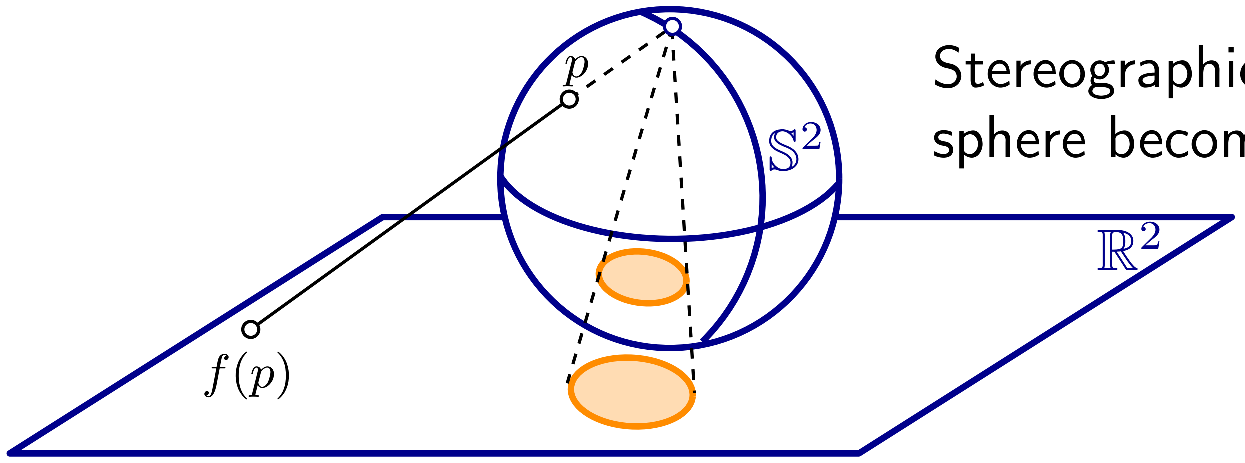


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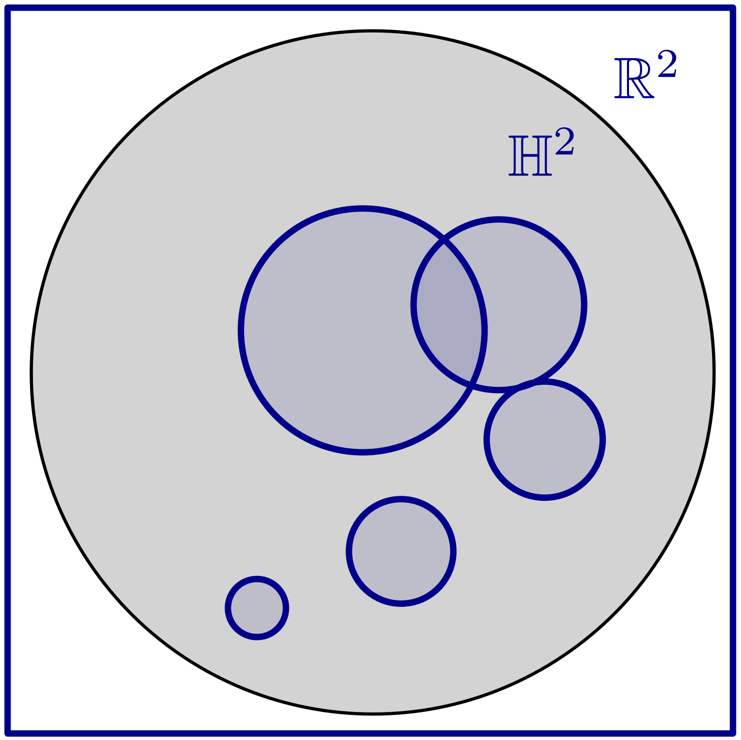

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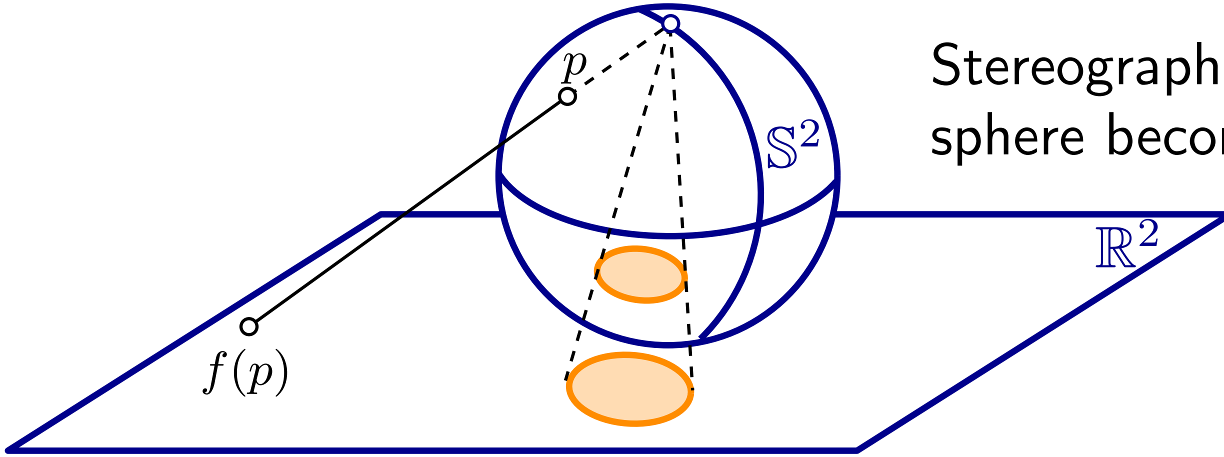


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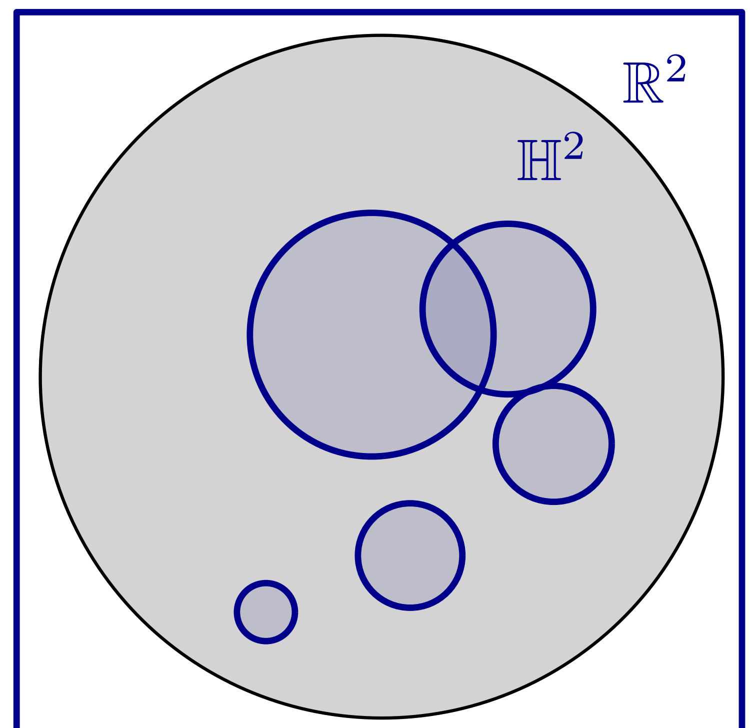
Sphere of radius $R$ (curvature $\kappa = 1/R^2$), with disks of radius $1$

$\Updownarrow$

Sphere of radius $1$ (curvature $1$), with disks of radius $r = 1/R$.

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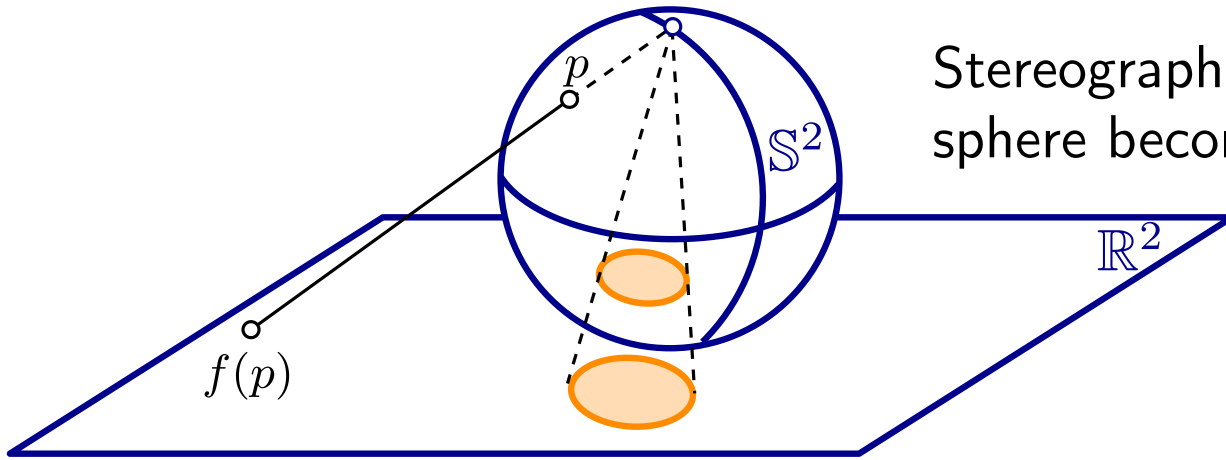


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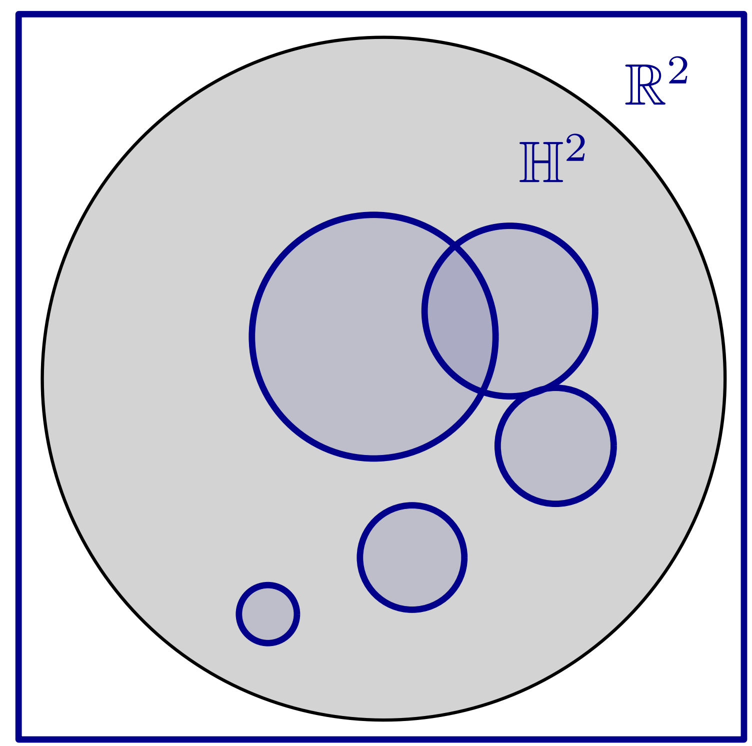


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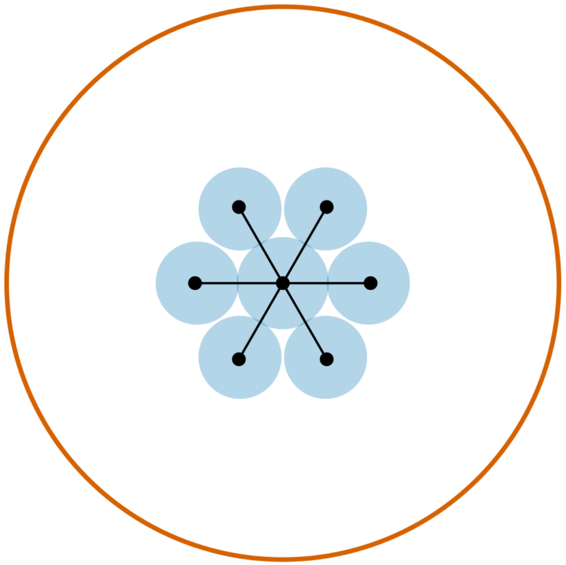
SUDG($r$) $\subsetneq$

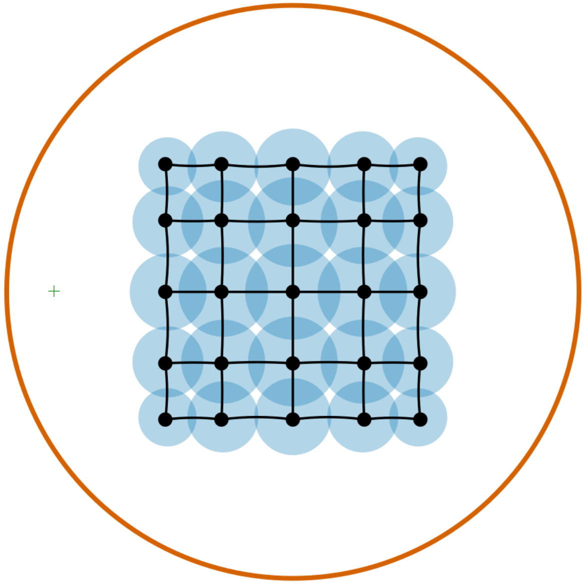
UDG $\subseteq$ DG

HUDG($r$) $\subsetneq$

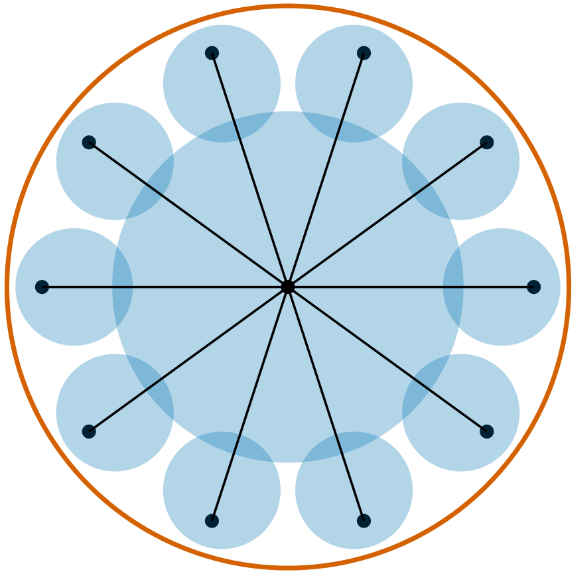
# The impact of radius on HUDG($r$)



$r = 1/\sqrt{n}$
almost-Euclidean



$r = \log n$
very hyperbolic,
tree-like



# Results on Independent Set in $\mathbb{H}^2$

**Theorem (NEW)**
Let $G \in \mathrm{HUDG}(r)$ and let $k \geq 0$. Then we can decide if there is an independent set of size $k$ in $G$ in $n^{O(1+\frac{1}{r}\log k)}$ time.

| $r = 1/\sqrt{k}$ | $r = 1$ | $r = \log k$ |
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*Ply* of disks: maximum # of overlapping disks at any point of $\mathbb{H}^2$.

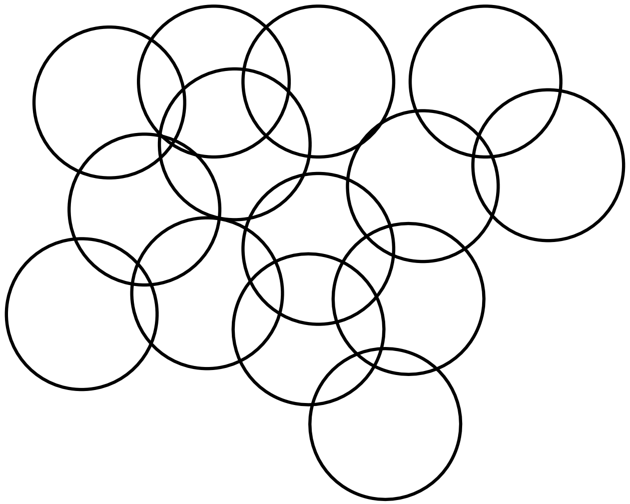
**Theorem (NEW)**
Let $\varepsilon \in (0,1)$ and let $G \in \mathrm{HUDG}(r)$ have ply $\ell$. Then a $(1-\varepsilon)$-approximate maximum independent set of $G$ can be found in
$O\left(n^4 \log n\right) + n \cdot \left(\frac{\ell}{\varepsilon}\right)^{O(1+\frac{1}{r}\log\frac{\ell}{\varepsilon})}$ time.

- quasi-polynomial in $1/\varepsilon$
- $\varepsilon = 1/n$, $\ell = n$ extends exact algo.

# Plan of attack: DP on noose hierarchy

# DP on solution separators

Guess separators for the (unkown) solution!



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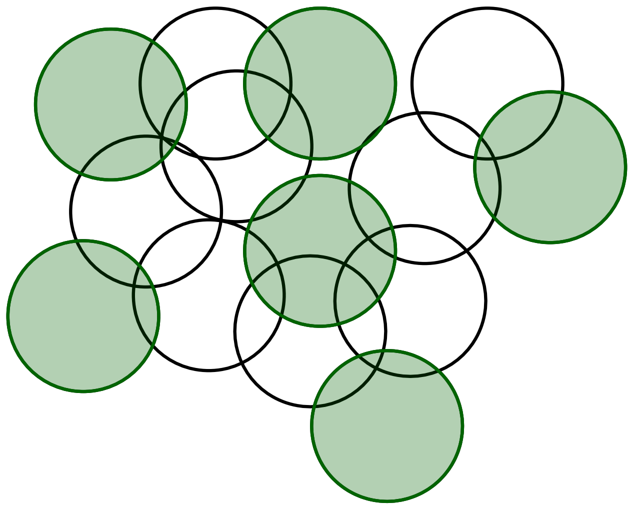

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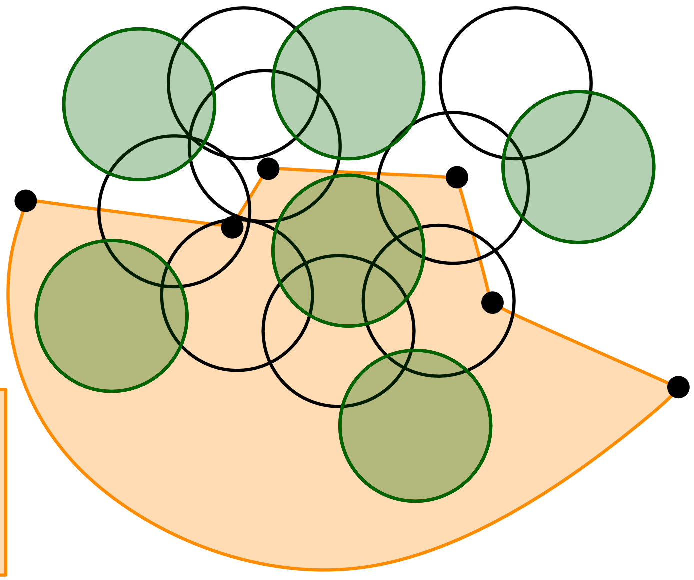

Closed curve ("noose") with $o(k)$ vertices, chosen from $\mathrm{poly}(n)$ possible curves.

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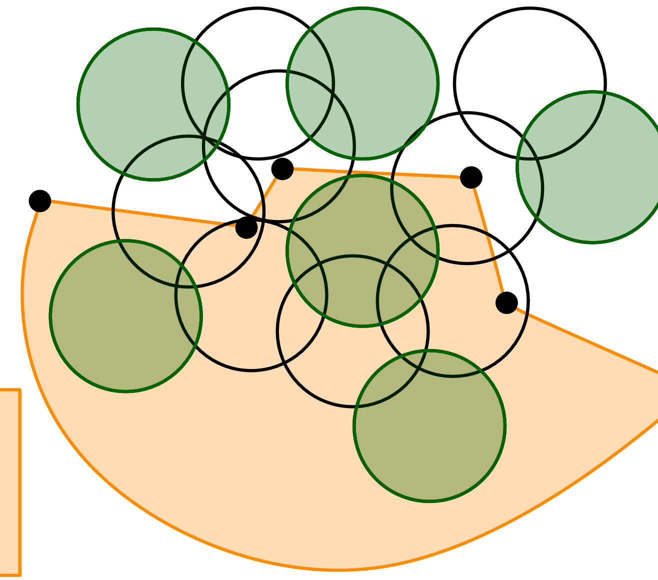

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**Theorem (NEW), oversimplified**
$G$ has indep. set of size $k$
$\Leftrightarrow$ there is a "well-spaced" hierarchy of $O(1 + \frac{\log k}{r})$ complexity nooses.

# Voronoi, Delaunay

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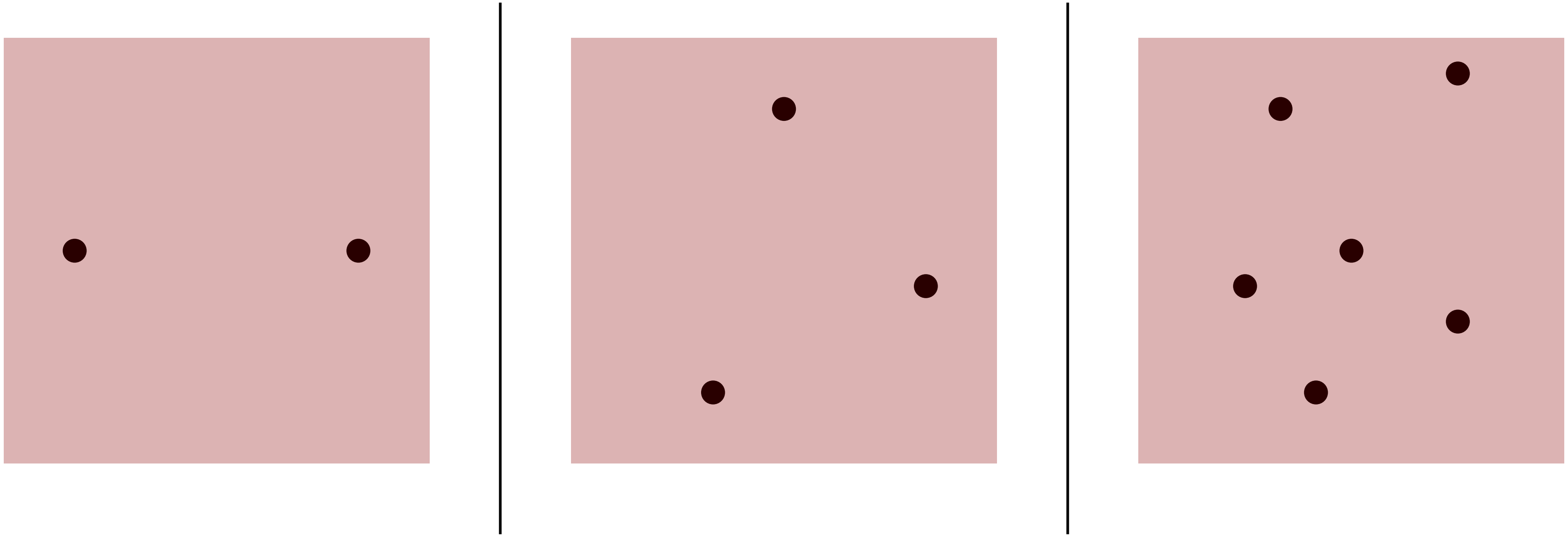

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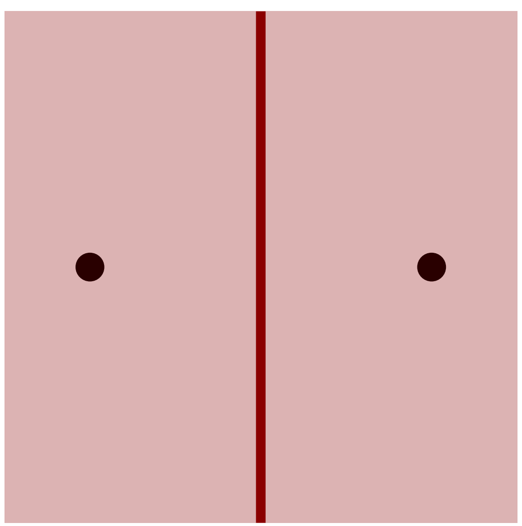
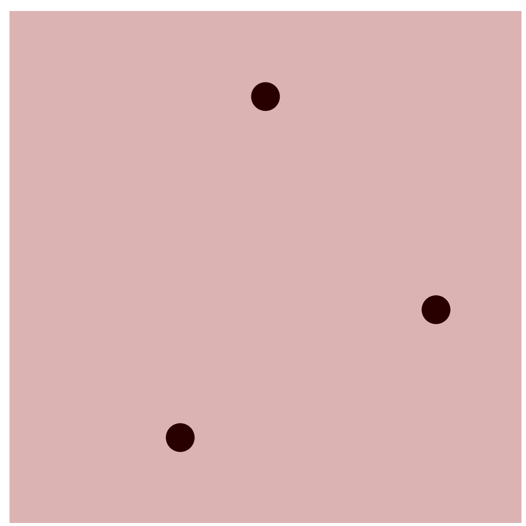
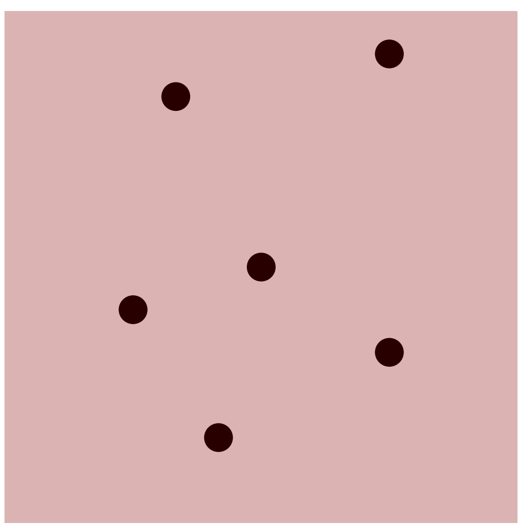


perpendicular bisector





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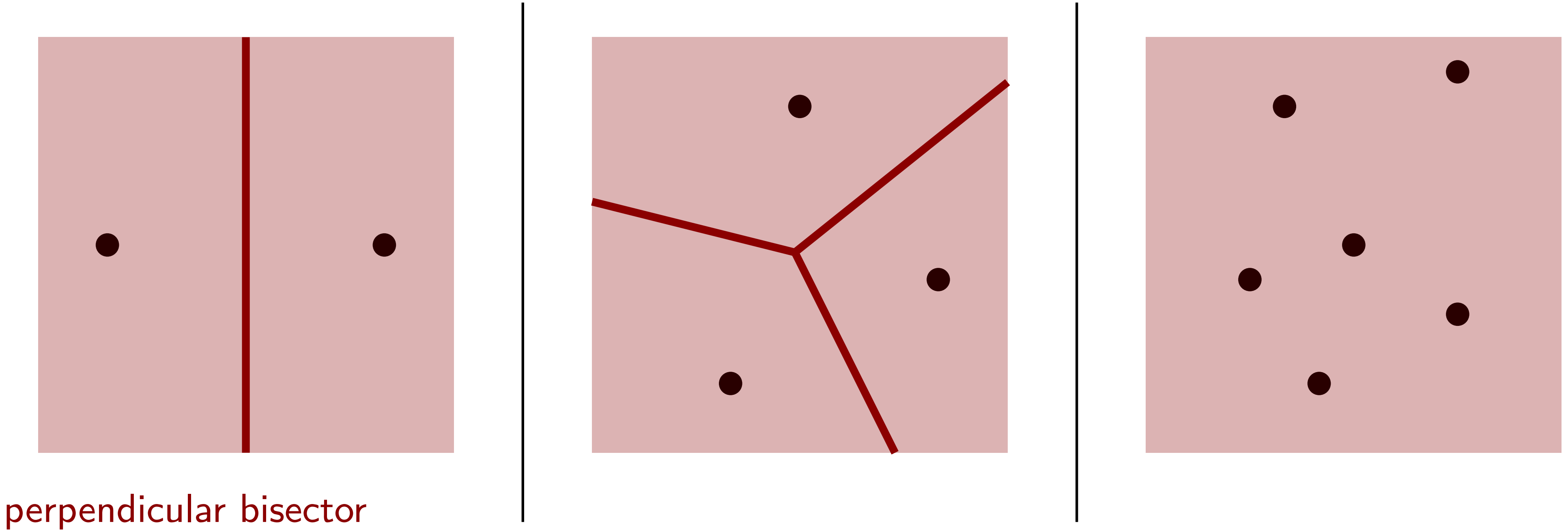

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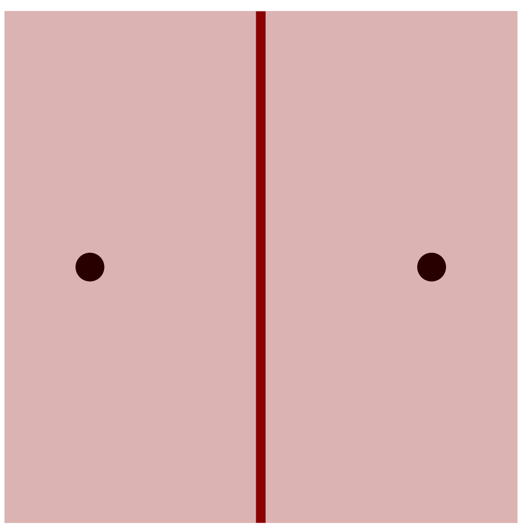

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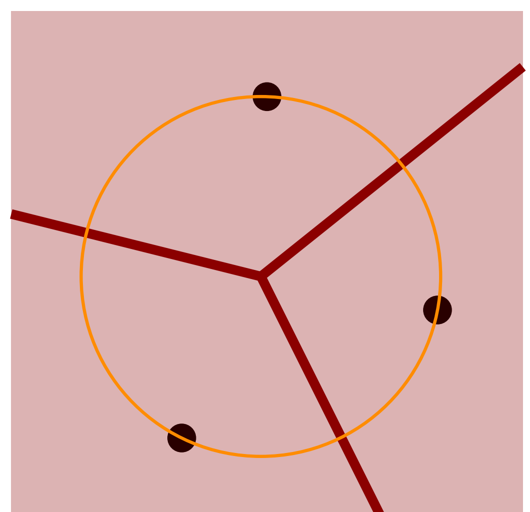
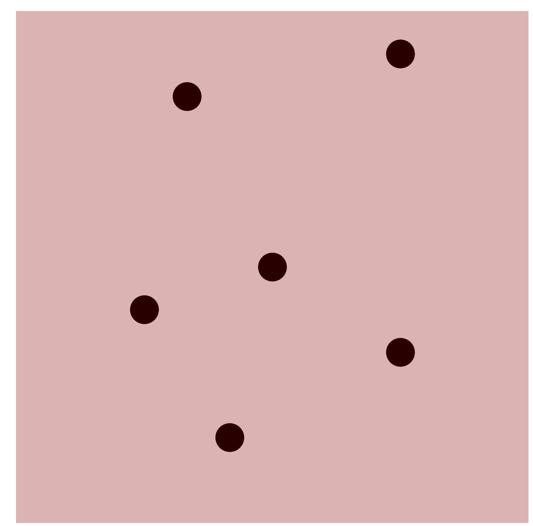


bisectors, center of circumcircle



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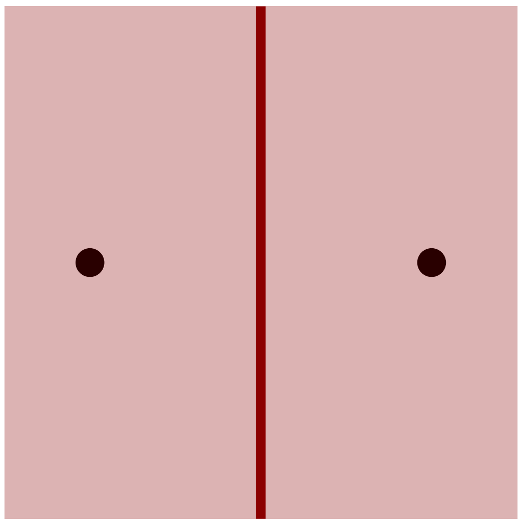

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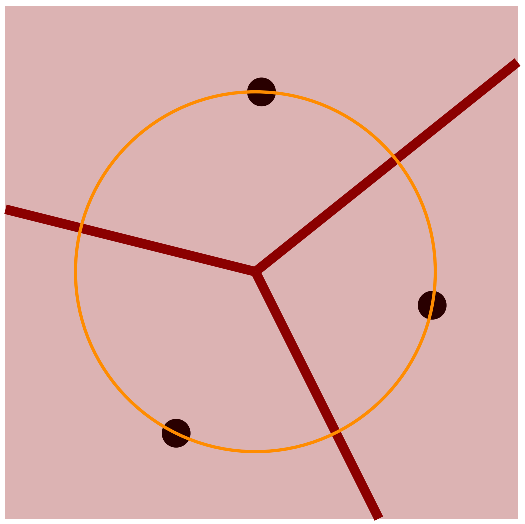

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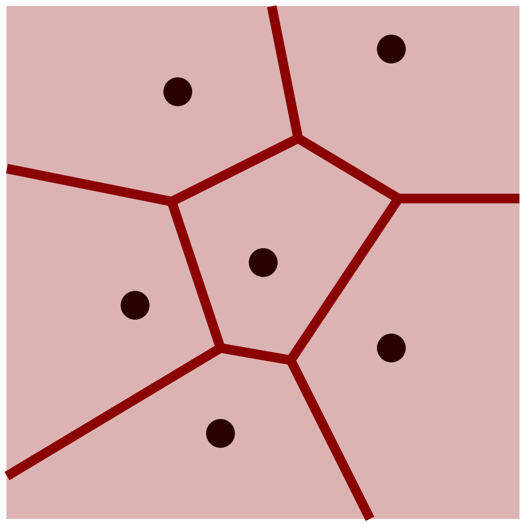
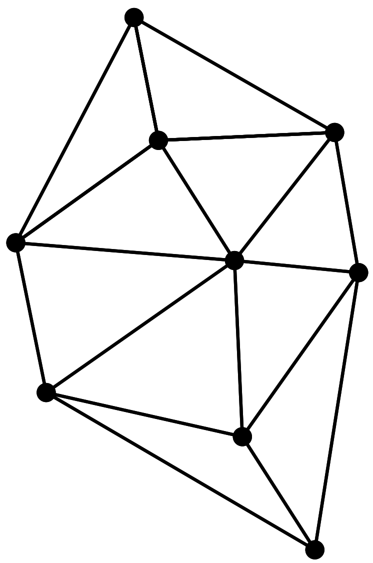


# Delaunay triangulations

Triangulation of $P$:
subdivision of $\text{conv}(P)$ into triangles whose vertex set is $P$

**Definition** A Delaunay triangulation of $P$ is a triangulation where the circumcircle of any triangle has no points of $P$ in its interior.



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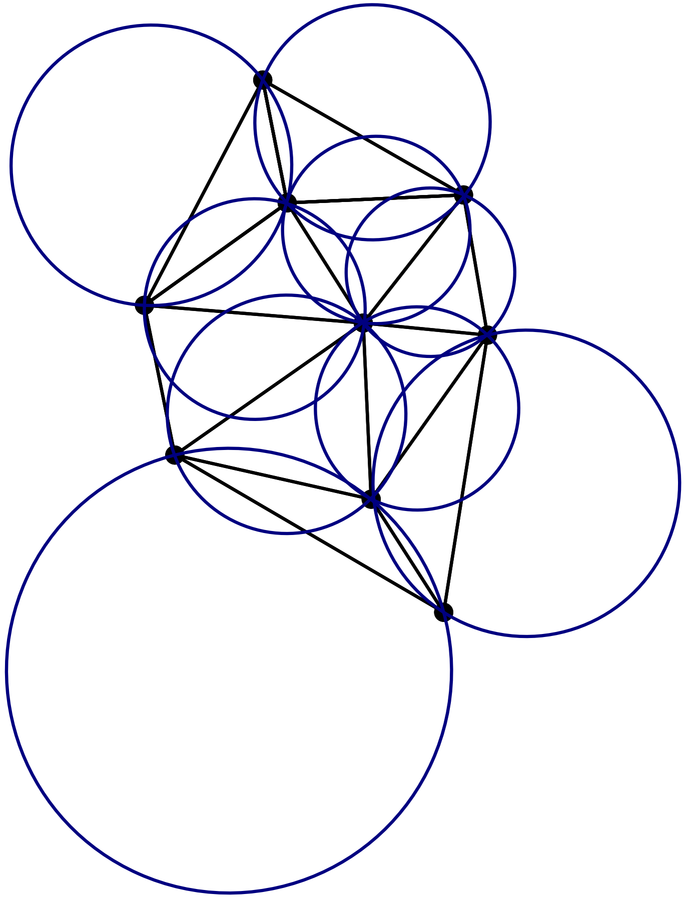

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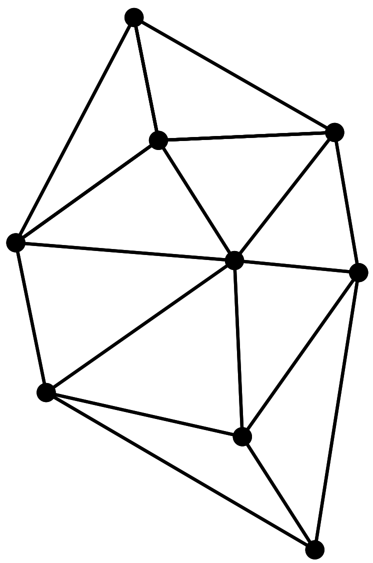
- dual of Voronoi diagram



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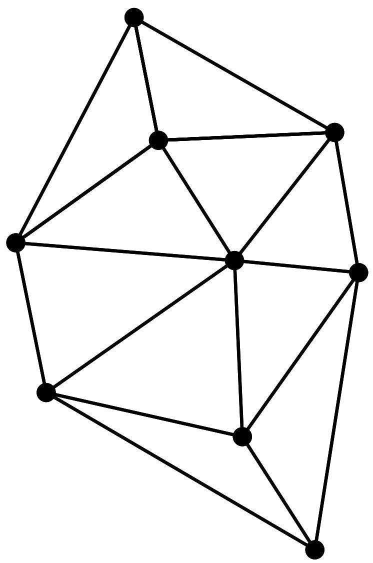


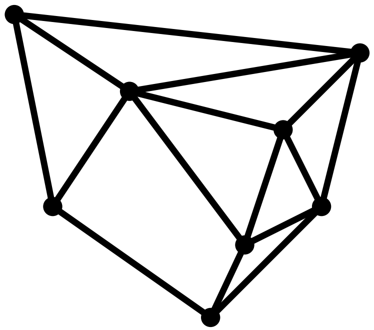
In $\mathbb{H}^2$ the Delaunay triangulation:

- is a triangulation (planar graph)
- can be comptued in $O(n \log n)$
- ~~can approximate distances on $P$~~
- dual of Voronoi diagram

# Outerplanarity

1-outerplanar: all vertices on outer face
k-outerplanar: removing vertices of outer face gives $(k-1)$-outerplanar

Outerplanarity $\simeq$ # of "vertex layers"



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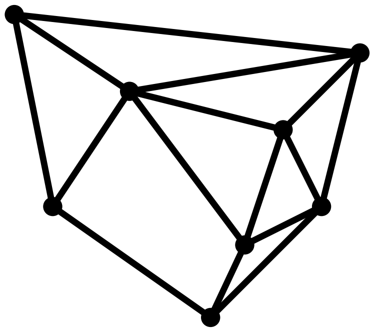

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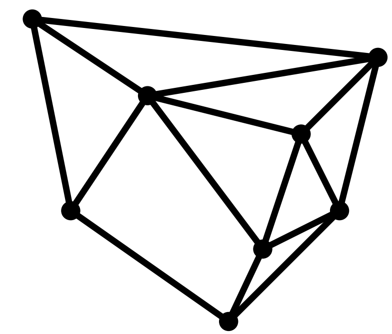

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Let $S$ be a set of $k$ points in $\mathbb{H}^2$ with pairwise distance at least $2r$. Then the Delaunay triangulation of $S$ is $1+O\left(\frac{\log k}{r}\right)$-outerplanar.

# Outerplanarity when $r > 1$ (sketch)

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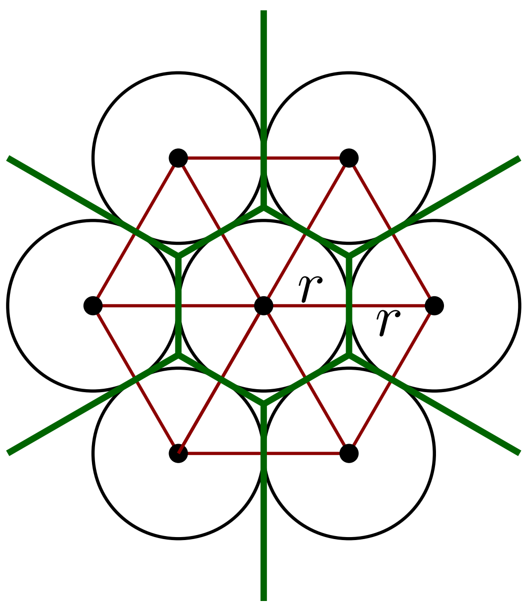
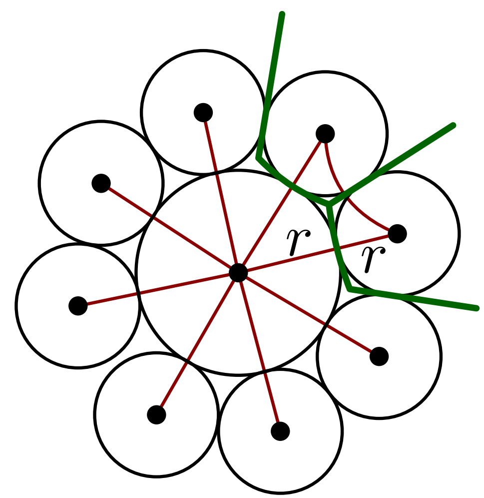
*Proof idea.* Since $r > 1$, the sudden expansion of $\mathbb{H}^2$ means that it is harder to "surround" a disk with disks.

Inner vertex of Delaunay $\Leftrightarrow$ bounded face of Voronoi.
Each Voronoi cell contains a radius $r$ disk.




Delaunay is a planar graph where inner vertices have degree $\geq e^r$!

But planar $\Rightarrow$ average degree $< 6 \Rightarrow$ at most $6k/e^r$ inner vertices.

# Conclusion

- Problem complexity can change when curvature changes.

- As curvature goes from $\kappa = 0$ to $\kappa = -\log^2 n$,
  Delaunay triangulation outerplanarity decreases,
  HUDG$_\kappa$ becomes separable with shorter nooses

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Let $G \in \mathrm{HUDG}(r)$ and let $k \geq 0$. Then we can decide if there is an independent set of size $k$ in $G$ in $n^{O(1+\frac{1}{r}\log k)}$ time.

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Disk graphs on hyperbolic surfaces?

Varying curvature (and genus) for spanners? visibility graphs?

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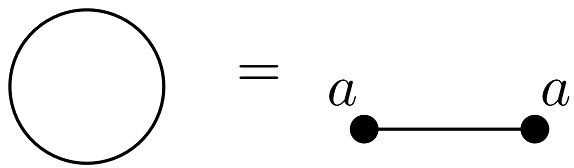
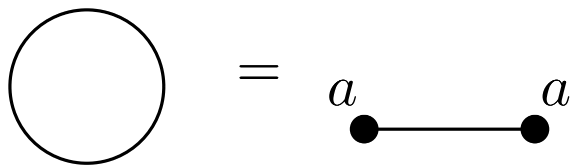
Thanks!

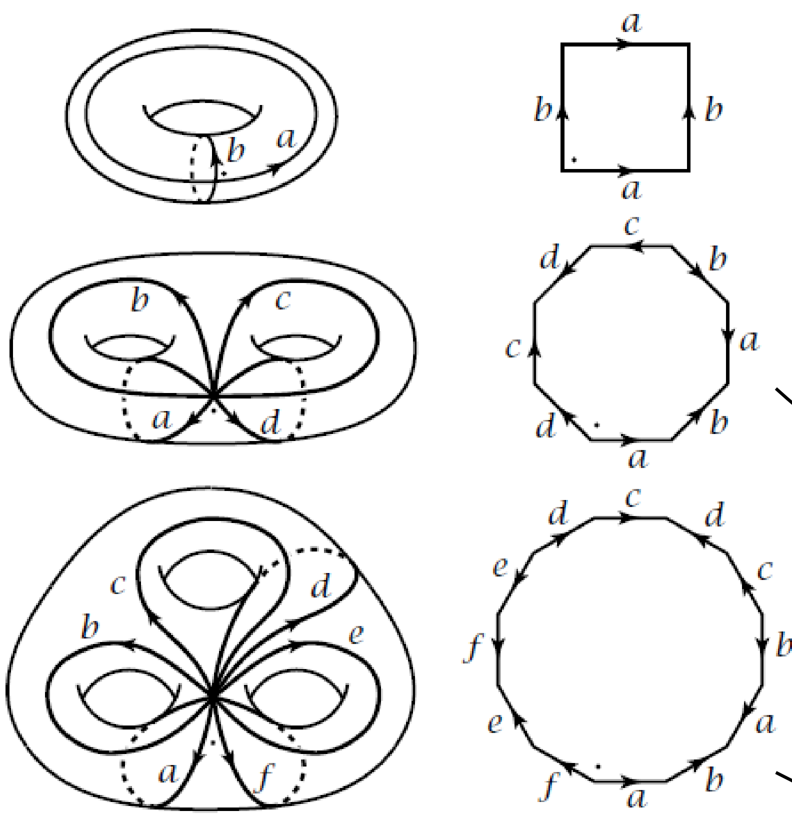
# Musings on hyperbolic surfaces

# Riemann coverings



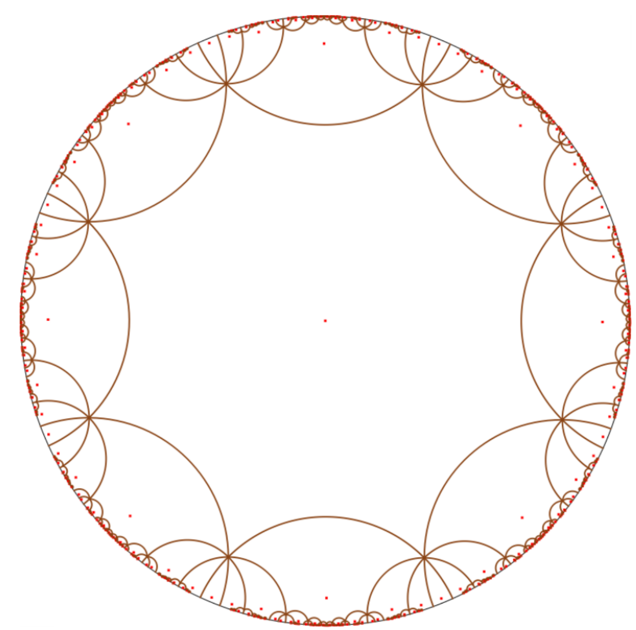
# Riemann coverings




Flat torus, curvature $= 0$

Bolza surface, curvature $= -1$



Regular hyperbolic $12$-gon with angles $\pi/6$.

Uniformization theorem $\Rightarrow$: when $g > 1$, then "natural" cover is hyperbolic!

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planar graph

↕

disk graph

genus g graph

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| planar graph | genus g graph | minor-free graph? |
|---|---|---|
| $\updownarrow$ | $\updownarrow$ | $\updownarrow$ ? |
| disk graph | $\mathbb{H}^2$ disk graph with period Schläfli $(4g, 4g)$ | $\mathbb{H}^2$ periodic disk graph |

# Line separator, simple divide and conquer

Strategy:

- find a point $p$ s.t. any line through $p$ has $\leq 2n/3$ disks on each side
- take a random line $\ell$ through $p$
- show that $r$-neighborhood of $\ell$ intersects small number of *cliques*
- guess $OPT \cap N(\ell, r)$, at most $1$ disk from each clique near $\ell$
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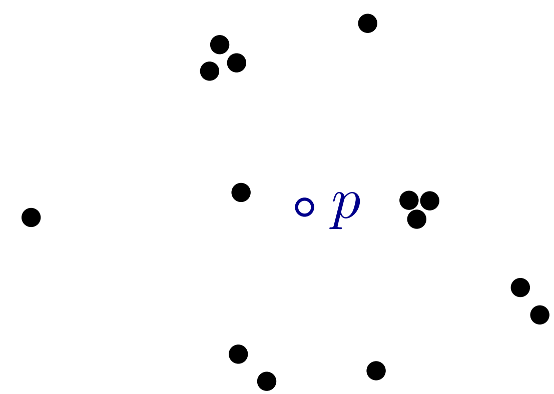

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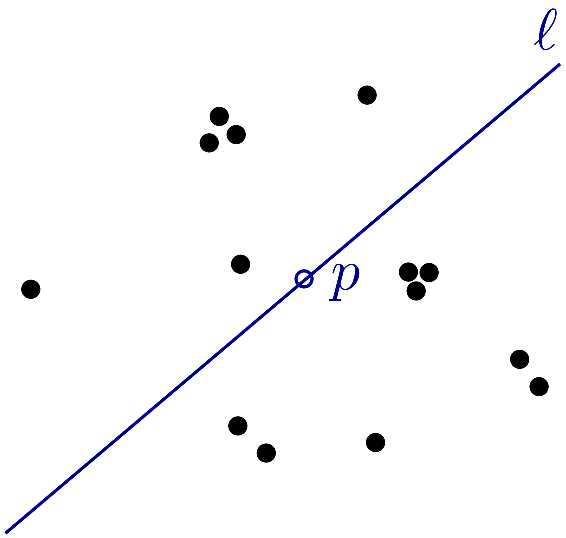

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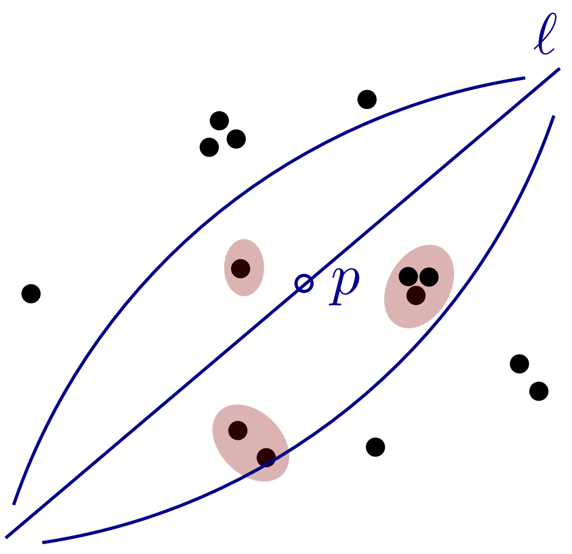

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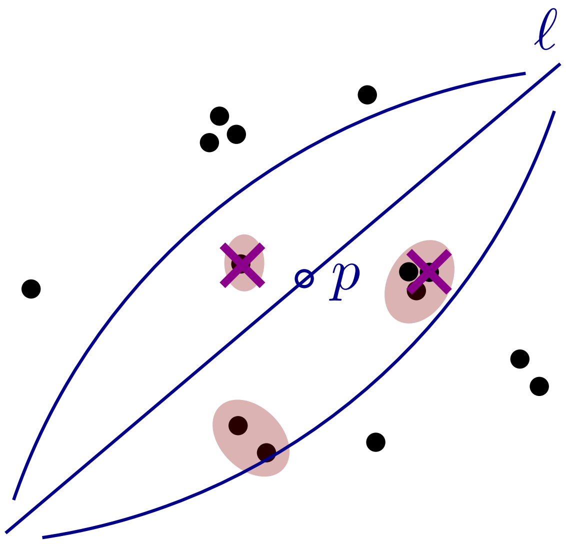

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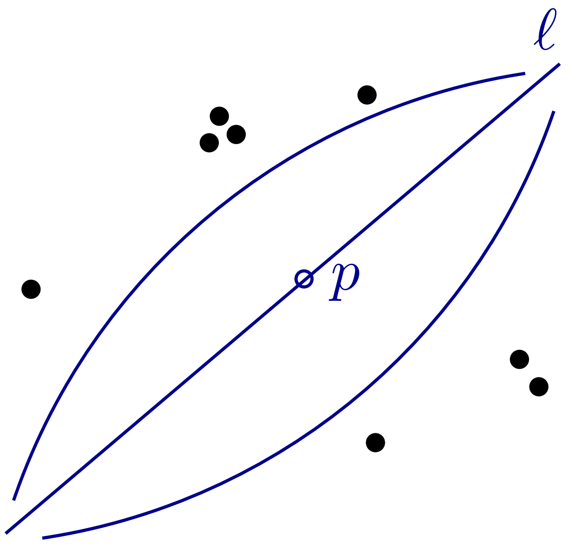

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**Theorem (NEW)**
Let $G \in \mathrm{HUDG}(r)$. Then $G$ has a separator $S$ that can be covered with $O\big(\log n \cdot \big(1 + \frac{1}{r}\big)\big)$ cliques, such that all conn. components of $G - S$ have at most $\frac{2}{3}n$ vertices.

when $r = \Omega(\log n)$, this yields quasi-polynomial algo. for Independent Set.

Nice, but not good enough!

# Spanners, planar and Steiner spanners

A $t$-spanner for $P \subset \mathbb{X}$ is a geometric graph $G$

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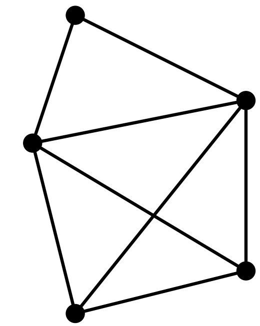

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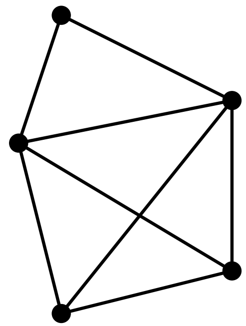

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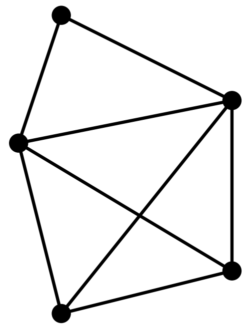

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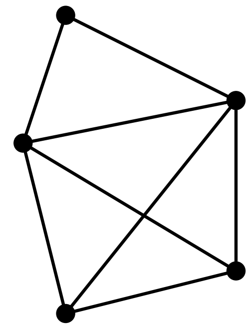

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A *Steiner spanner* adds Steiner points $S \subset \mathbb{X}$

- $P \cup S$ are the vertices
- Only approximates distances among $P$