Distributed Quantum Advantage in Locally Checkable Labellings

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Joint work with:
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Distributed Quantum Workshop 2025

Distributed Quantum Advantage in LCLs

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- **Distributed**
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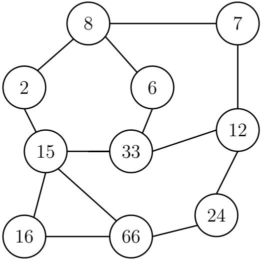
High level idea:

- **Distributed**
  the models of computations are LOCAL and its generalizations
- **Quantum**
  among those generalizations there is quantum-LOCAL
- **Advantage**
  the goal is separating quantum-LOCAL from the rest
- **Locally Checkable Labelings**
  ideally for LCLs

Definitions: LOCAL

LOCAL

- The network is a graph $G = (V, E)$
- Nodes have input and unique IDs
- IDs belong to $\{1, 2, \ldots, n^c\}$
- Computation is **unbounded**, but terminating
- Message size is **unbounded**



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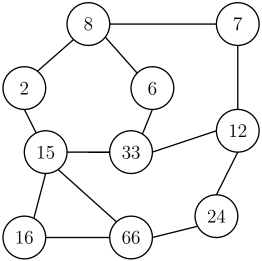

In **one communication round**, every node $v \in V$ performs the following:

- send messages to all neighbours in $\mathcal{N}_1(v)$
- receive messages from all neighbours in $\mathcal{N}_1(v)$
- update internal state, and possibly stop

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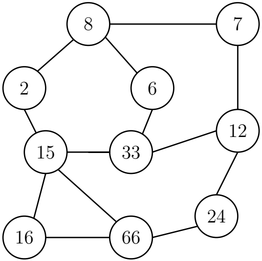

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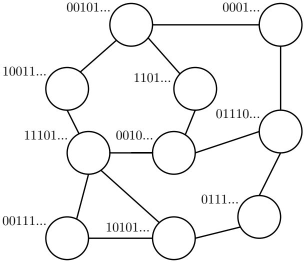
Complexity:
Number of communication rounds as a function of n before every node stops

Definitions: rand-LOCAL

rand-LOCAL with **private randomness**

- The network is a graph $G = (V, E)$
- Nodes have input and a **random binary string**
- No IDs, but they can be guessed w.h.p.
- The rest is the same as LOCAL



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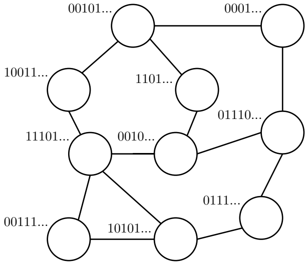

rand-LOCAL with **shared randomness**

- all of the above
- one addition random binary string **known to every node**

Definitions: quantum-LOCAL

quantum-LOCAL with **private entanglement**

- The network is a graph $G = (V, E)$
- Nodes have input and **access to qubits**
- Computation is any unitary U of **finite dimensions**
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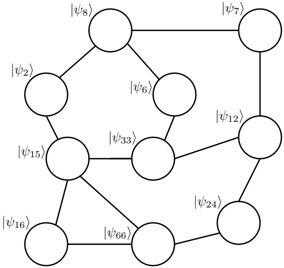

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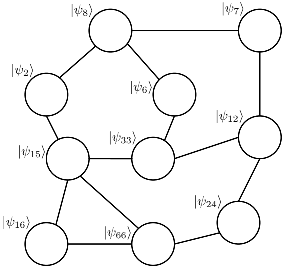
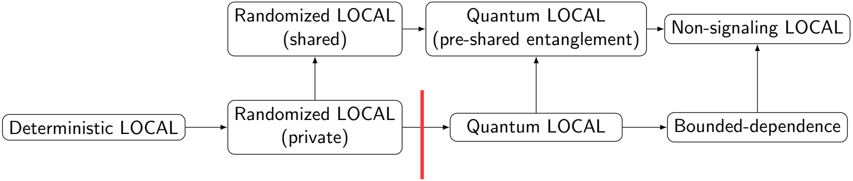


quantum-LOCAL with **pre-shared entanglement**

- all of the above
- **before** the computation starts, a set Q of qubits is prepared in any desired quantum state
- **after** the computation starts, every node has access to a distinct subset of Q

Distributed Landscape



Today: latest separation between quantum-LOCAL and rand-LOCAL (private)

Locally Checkable Labellings

We want quantum advantage for Locally Checkable Labellings (LCLs) [Naor and Stockmeyer - STOC '93]

- An LCL defines a labelling problem
- Allowed configurations defined as a **set of neighbourhoods**
- Checking radius r is constant

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- An LCL defines a labelling problem
- Allowed configurations defined as a **set of neighbourhoods**
- Checking radius r is constant

Checking radius r: if every node $v \in V$ sees a correct labelling in its neighbourhood $\mathcal{N}_r(v)$, then the labelling is globally correct.

Locally Checkable Labellings

Why LCLs? Many reasons, for example:

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- **Large enough** class of problems to capture interesting problems
- e.g. colouring, max matching, MIS, ...
- **Small enough** class of problems to prove something interesting
- e.g. no complexities between $O(\log n)$ and $O(\log *n)$

Prior Quantum Advantage

Previous result on LOCAL vs quantum-LOCAL
[Le Gall, Nishimura, Rosmanis - STACS '19]

- There exists a problem where quantum-LOCAL **beats** LOCAL
- quantum-LOCAL complexity: $O(1)$
- LOCAL complexity: $\Omega(n)$
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Consider graphs of maximum degree $\Delta \geq 3$.
There exists a family of LCL problems $\mathcal{P}_\Delta$ such that:
[Balliu et al. - STOC '25]

- $\mathcal{P}_\Delta$ has checking radius $O(1)$ for any Δ
- Solving $\mathcal{P}_\Delta$ takes $O(1)$ in quantum-LOCAL
- Solving $\mathcal{P}_\Delta$ takes $\Omega(\Delta)$ rand-LOCAL

Our Results

There exists an LCL problem $\mathcal{P}$ such that:
[Balliu et al. - SODA '26]

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We actually prove this for finitely dependent distributions

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In a GHZ game there are three players A, B and C.
They can share information beforehand, then communication stops.

Given bits (x, y, z) they must output (a, b, c) such that:

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Namely:

- input $= (0, 0, 0) \Rightarrow$ output has even parity
- input $\neq (0, 0, 0) \Rightarrow$ output has odd parity

x	y	z	$a \oplus b \oplus c$
0	0	0	0
1	1	0	1
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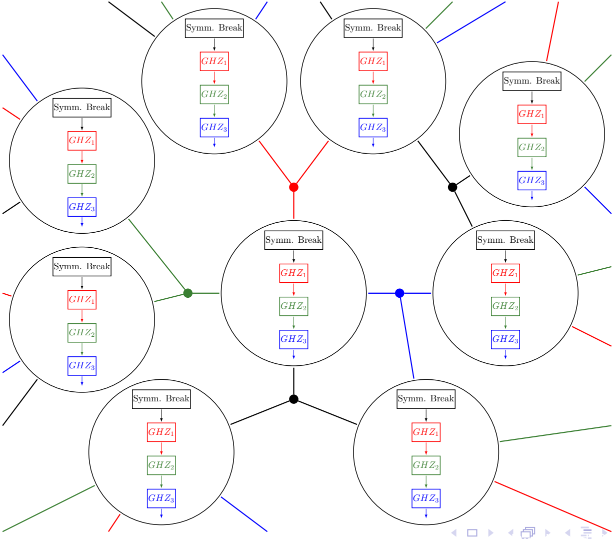
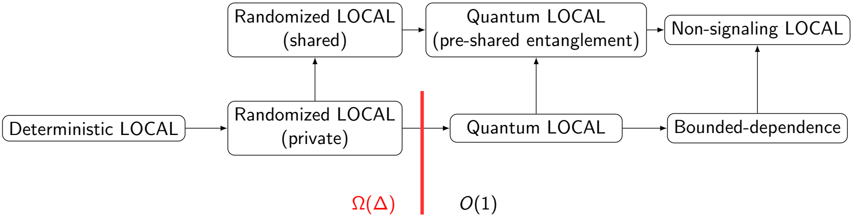
$\Rightarrow$ this **does not** show quantum advantage

Iterated GHZ

For an advantage in quantum-LOCAL, we need to play multiple games



Quantum Advantage as a function of Δ



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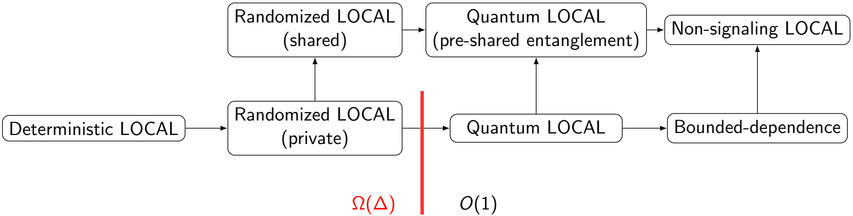

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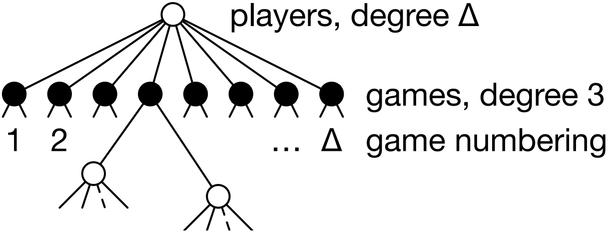
Why a *family* of LCLs? Because fixing one LCL fixes Δ.

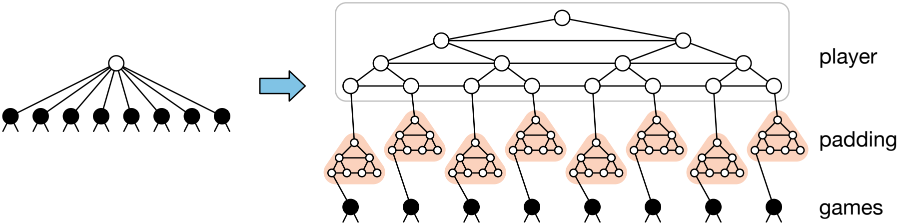
Padding

We are starting from this structure



Padding

To improve this result, we "pad" active nodes



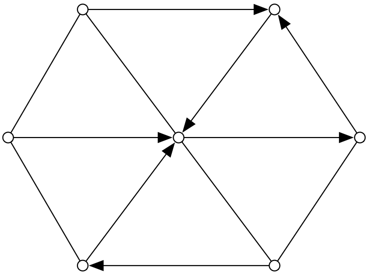
This way, we can achieve the complexity that we want by borrowing results from previous work [Balliu et al. - PODC '20]

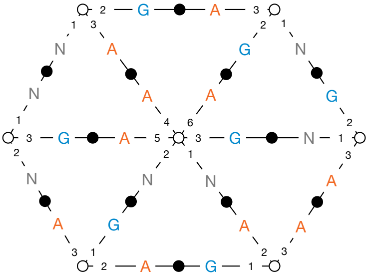
The important property to check is that the LCL must be *linearizable*

Padding

Example of linearizable LCL: edge grabbing

Every node grabs one edge, which must be the only one oriented way from it



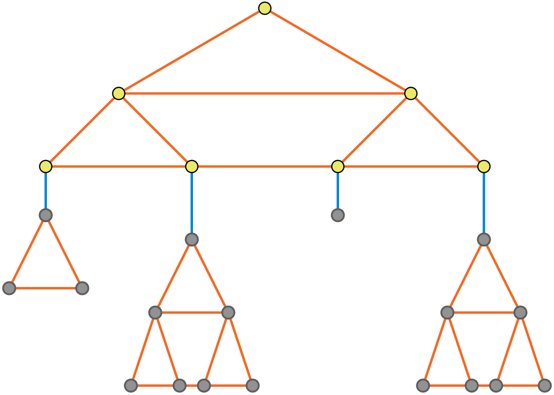


G : grabbed
N : not grabbed
A : already grabbed a previous edge

Linearizable: labelling (N,...,N,G,A,...,A) can be check by just looking at predecessor and successor

Padding

Back to our LCL, recall that iterated-GHZ requires the output of a game to be the input for the next one

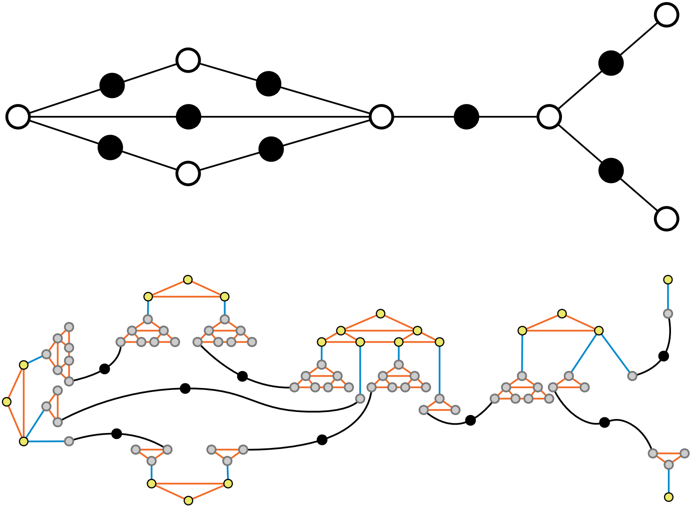


The path connecting the yellow "leaves" allows to *linearize* iterated-GHZ.

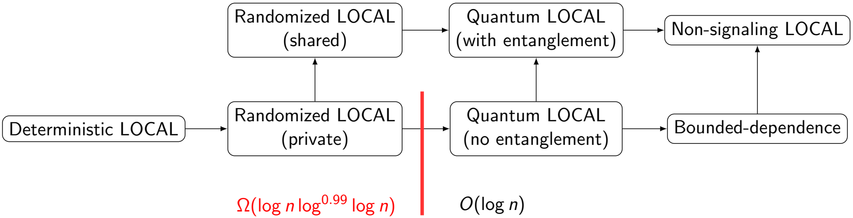
Moreover, now $\Delta = 4$ and the depth of each pyramid is $O(\log n)$.

Padding

Final construction



Quantum Advantage as a Function of n



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Limits to Quantum Advantage

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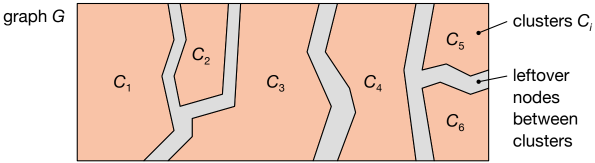
Limits to Quantum Advantage

Consider a T-dependent distribution

We cannot just sample from it blindly

- nodes at distance less than T are dependent

Our approach: we cluster the graph



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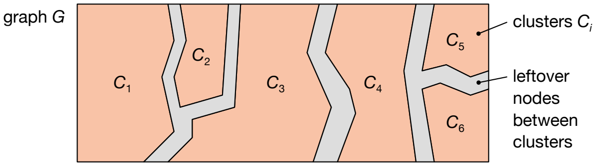
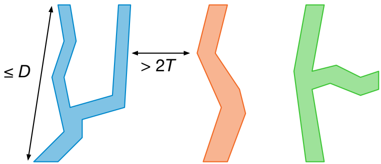


We organize the leftover nodes into partitions at distance $\Omega(T)$



Maximum diameter for a partition: $D = O(T \cdot \sqrt{n/T}) = O(\sqrt{nT})$

Reason: Scout up to distance $2T$, repeat if unclustered node found

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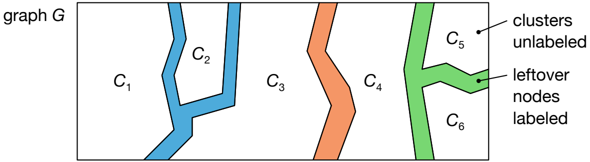
Now we can sample in every partition independently in time $O(\sqrt{nT})$



Now complete what's in the clusters by brute force in time $O(\sqrt{nT})$

Total time: $O(\sqrt{nT})$

- Clustering: $O(\sqrt{nT})$
- Partitioning and sampling: $O(\sqrt{nT})$
- Completing by brute force: $O(\sqrt{nT})$

Summary

Take home messages:

Type of problem	quantum-LOCAL	rand-LOCAL	Publication
global	$O(1)$	$O(n)$	STACS '19
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