NPA hierarchy

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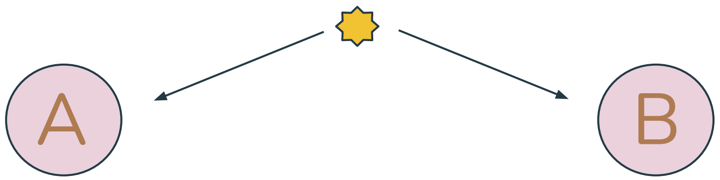
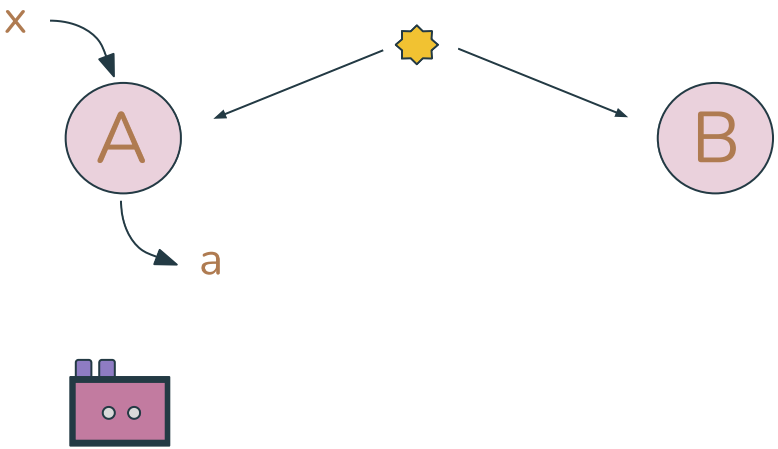
A quantum theory of information



A quantum theory of information



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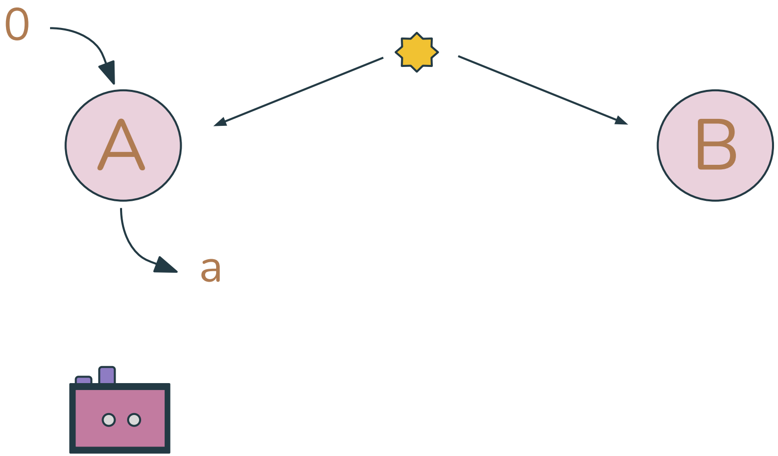

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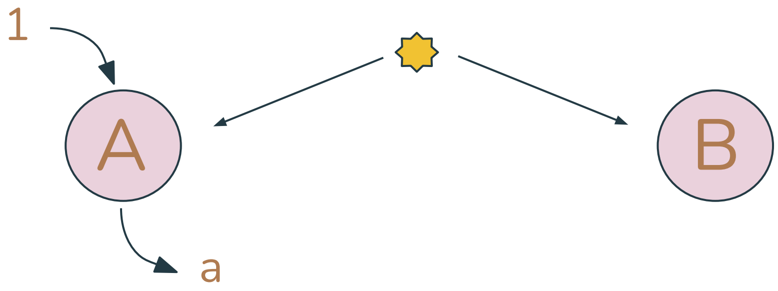

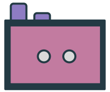

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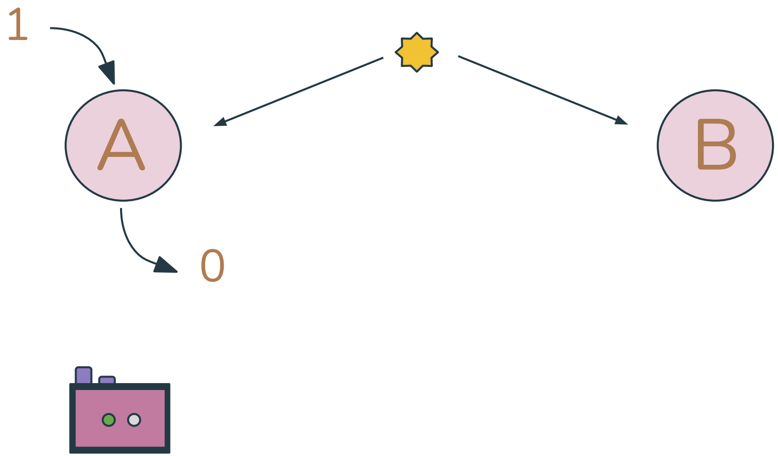
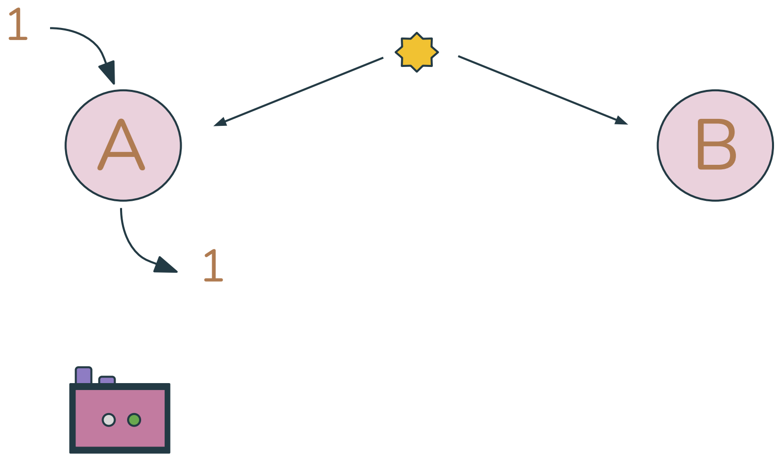


A quantum theory of information



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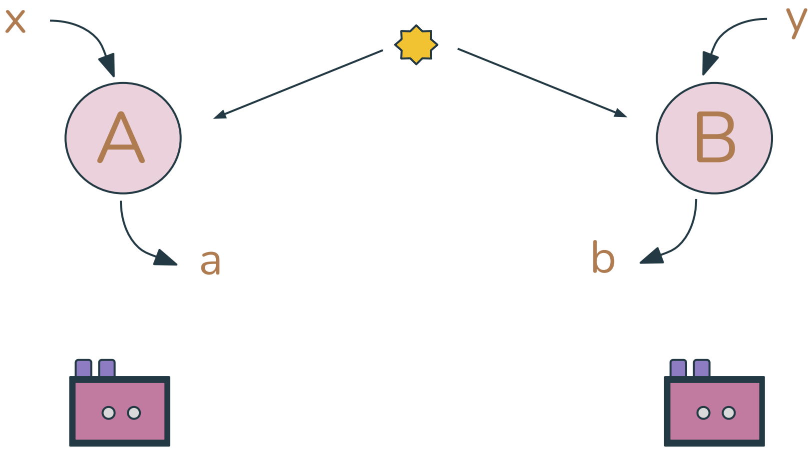

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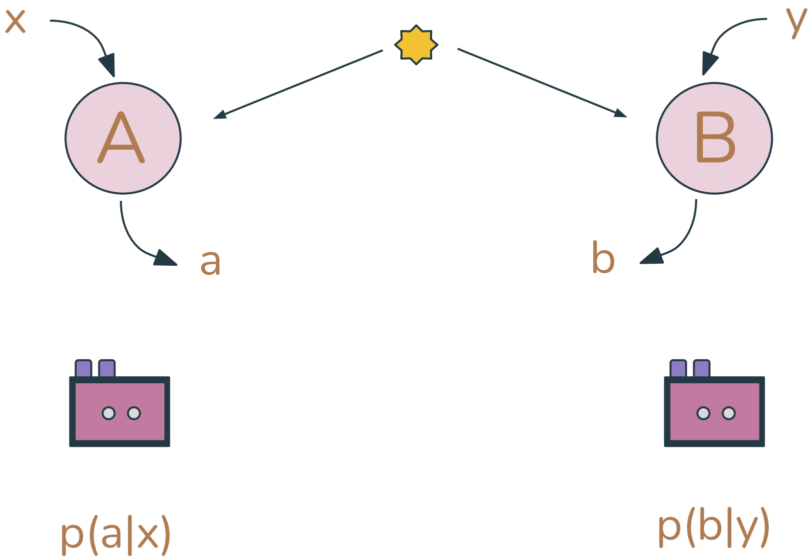

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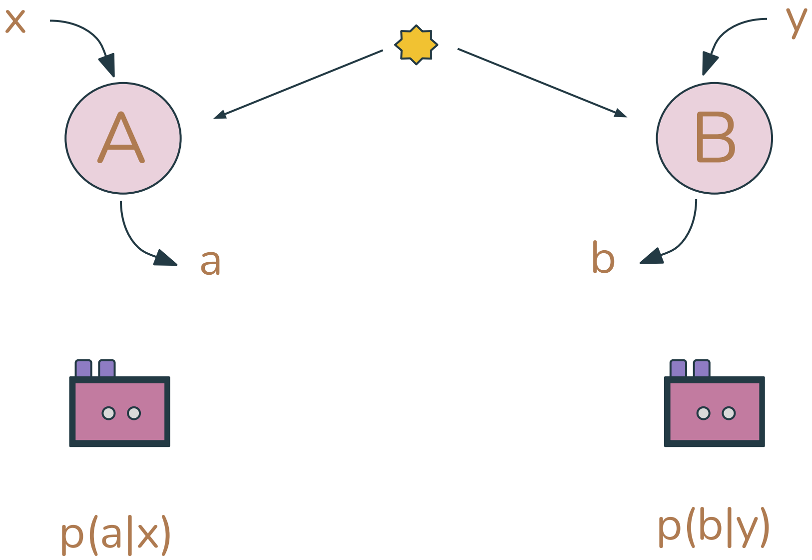
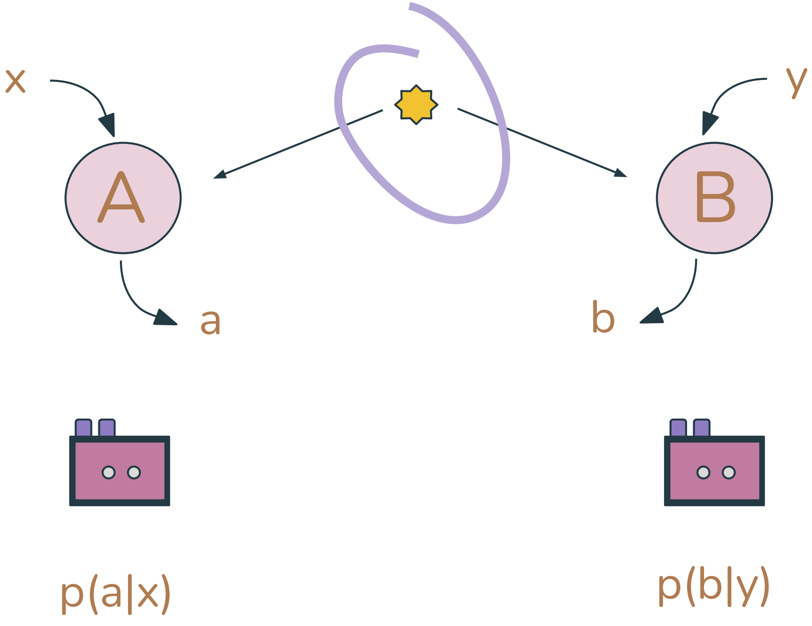


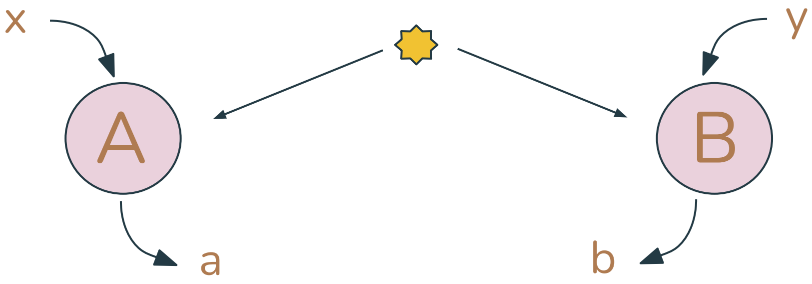
p(ab|xy)

A quantum theory of information

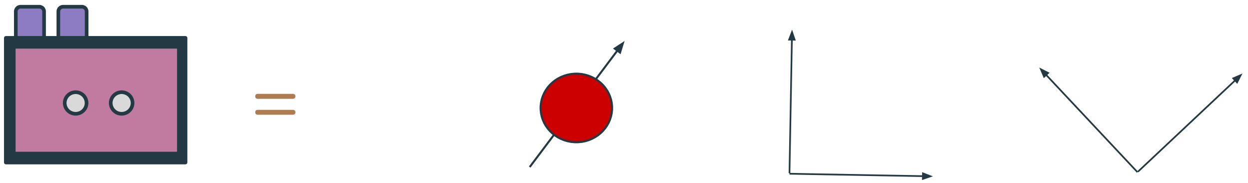


p(ab|xy)

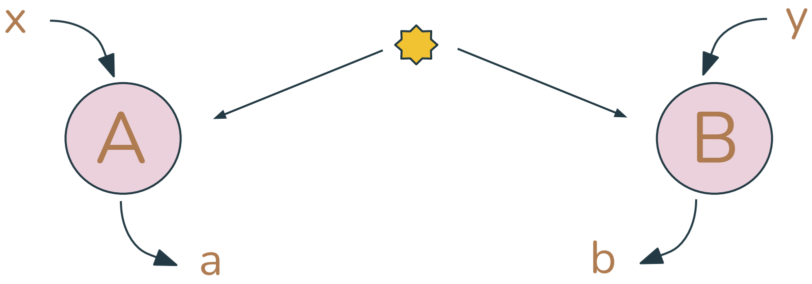
A quantum theory of information



p(ab|xy)

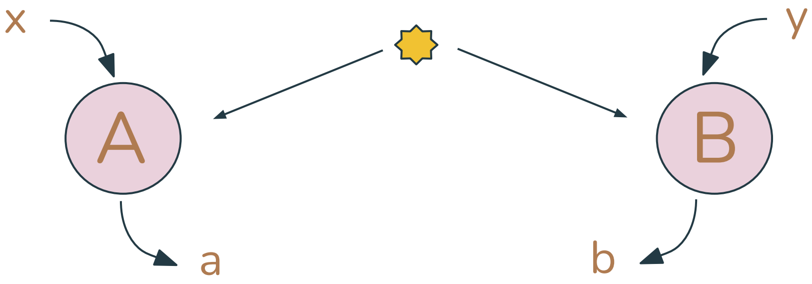


A quantum theory of information



$$p(ab|xy) = Tr[\Psi A_a^x B_b^y]$$

A quantum theory of information



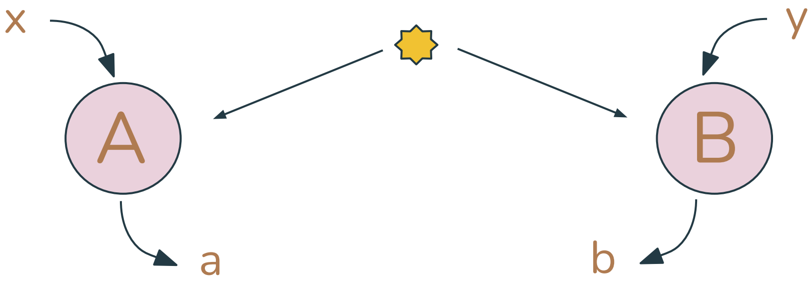
$$p(ab|xy) = Tr[\Psi A_a^x B_b^y]$$

state

Bob's measurements

Alice's measurements

A quantum theory of information



$$\mathcal{M} = \{M_a\}_a$$
$$\sum_a M_a = \mathbb{1}$$
$$M_a \geq 0, \quad \forall a$$

$$p(ab|xy) = Tr[\Psi A_a^x B_b^y]$$

$$Tr[\Psi] = 1$$
$$\Psi \geq 0$$

state

Bob's measurements

Alice's measurements

this is a feasibility problem!

$$\begin{aligned} &given \quad p(ab|xy) \\ &find \quad \Psi,\ \{A_a^x\},\ \{B_b^y\} \\ &s.t. \quad p(ab|xy) = Tr[\Psi A_a^x B_b^y] \quad \forall a, b, x, y \\ &\qquad\ \Psi \text{ is a state} \\ &\qquad\ \{A_a^x\},\ \{B_b^y\} \text{ are measurements} \end{aligned}$$

this is a feasibility problem!

$$\begin{aligned}
&given && p(ab|xy) \\
&find && \Psi,\ \{A_a^x\},\ \{B_b^y\} \\
&s.t. && p(ab|xy) = Tr[\Psi A_a^x B_b^y] \quad \forall a, b, x, y \\
& && \Psi \text{ is a state} \\
& && \{A_a^x\},\ \{B_b^y\} \text{ are measurements}
\end{aligned}$$

but not an efficient one :(

SDPs are efficient problems semidefinite programs

$$
\begin{aligned}
\textbf{maximize} \quad & \mathrm{Tr}[AX] \\
\textbf{subject to} \quad & \Lambda_i(X) = B_i, \quad i = 1, \ldots, m \\
& \Gamma_j(X) \leq C_j, \quad j = 1, \ldots, n \\
& X \geq 0\,.
\end{aligned}
$$

$$
\begin{aligned}
given \quad & p(ab|xy) \\
find \quad & \Psi,\ \{A_a^x\},\ \{B_b^y\} \\
s.t. \quad & p(ab|xy) = Tr[\Psi A_a^x B_b^y] \quad \forall a, b, x, y \\
& \Psi \text{ is a state} \\
& \{A_a^x\},\ \{B_b^y\} \text{ are measurements}
\end{aligned}
$$

NOT SDP

not all is lost!

NPA hierarchy 1 inefficient problem $\Rightarrow$ ∞ efficient problems

not all is lost!

NPA hierarchy 1 inefficient problem $\longrightarrow$ ∞ efficient problems

if **any** of these problems is infeasible,
then p(ab|xy) cannot be quantum

The NPA hierarchy

Navascués, Pironio, Acín

suppose there exist state and measurements explaining p(ab|xy)

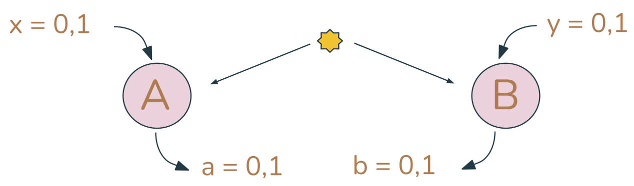
$$S_1 = \{\, \mathbb{1} \,\} \cup \{A_a^x\} \cup \{B_b^y\}$$

The NPA hierarchy

Navascués, Pironio, Acín

suppose there exist state and measurements explaining p(ab|xy)

$$S_1 = \{ \mathbb{1} \} \cup \{A_0^0, A_0^1, A_1^0, A_1^1\} \cup \{B_0^0, B_0^1, B_1^0, B_1^1\}$$



The NPA hierarchy

Navascués, Pironio, Acín

suppose there exist state and measurements explaining p(ab|xy)

$$|\Psi\rangle\,,A_0^0|\Psi\rangle\,,A_0^1|\Psi\rangle\,,A_1^0|\Psi\rangle\,,A_1^1|\Psi\rangle\,,B_0^0|\Psi\rangle\,,B_0^1|\Psi\rangle\,,B_1^0|\Psi\rangle\,,B_1^1|\Psi\rangle$$

The NPA hierarchy

Navascués, Pironio, Acín

suppose there exist state and measurements explaining p(ab|xy)

$$|\Psi\rangle\,,\, A_0^0|\Psi\rangle\,,\, A_0^1|\Psi\rangle\,,\, A_1^0|\Psi\rangle\,,\, A_1^1|\Psi\rangle\,,\, B_0^0|\Psi\rangle\,,\, B_0^1|\Psi\rangle\,,\, B_1^0|\Psi\rangle\,,\, B_1^1|\Psi\rangle$$

$$\Gamma_{ij} = \langle\Psi|O_i^\dagger O_j|\Psi\rangle, \quad O_i, O_j \in S_1$$

$$\Gamma_{00} = \langle\Psi|\Psi\rangle = 1$$

$$\Gamma_{26} = \langle\Psi|A_0^{0\dagger} B_0^1|\Psi\rangle = p(00|01)$$

Moment matrix

if p(ab|xy) is quantum, then

$$
\Gamma_1 = \begin{bmatrix}
\langle \Psi|\Psi\rangle & \langle A_0^0\rangle & \langle A_0^1\rangle & \langle A_1^0\rangle & \langle A_1^1\rangle & \langle B_0^0\rangle & \langle B_0^1\rangle & \langle B_1^0\rangle & \langle B_1^1\rangle \\
\langle A_0^0\rangle & \langle A_0^0A_0^0\rangle & \langle A_0^0A_0^1\rangle & \langle A_0^0A_1^0\rangle & \langle A_0^0A_1^1\rangle & \langle A_0^0B_0^0\rangle & \langle A_0^0B_0^1\rangle & \langle A_0^0B_1^0\rangle & \langle A_0^0B_1^1\rangle \\
\langle A_0^1\rangle & \langle A_0^1A_0^0\rangle & \langle A_0^1A_0^1\rangle & \langle A_0^1A_1^0\rangle & \langle A_0^1A_1^1\rangle & \langle A_0^1B_0^0\rangle & \langle A_0^1B_0^1\rangle & \langle A_0^1B_1^0\rangle & \langle A_0^1B_1^1\rangle \\
\langle A_1^0\rangle & \langle A_1^0A_0^0\rangle & \langle A_1^0A_0^1\rangle & \langle A_1^0A_1^0\rangle & \langle A_1^0A_1^1\rangle & \langle A_1^0B_0^0\rangle & \langle A_1^0B_0^1\rangle & \langle A_1^0B_1^0\rangle & \langle A_1^0B_1^1\rangle \\
\langle A_1^1\rangle & \langle A_1^1A_0^0\rangle & \langle A_1^1A_0^1\rangle & \langle A_1^1A_1^0\rangle & \langle A_1^1A_1^1\rangle & \langle A_1^1B_0^0\rangle & \langle A_1^1B_0^1\rangle & \langle A_1^1B_1^0\rangle & \langle A_1^1B_1^1\rangle \\
\langle B_0^0\rangle & \langle B_0^0A_0^0\rangle & \langle B_0^0A_0^1\rangle & \langle B_0^0A_1^0\rangle & \langle B_0^0A_1^1\rangle & \langle B_0^0B_0^0\rangle & \langle B_0^0B_0^1\rangle & \langle B_0^0B_1^0\rangle & \langle B_0^0B_1^1\rangle \\
\langle B_0^1\rangle & \langle B_0^1A_0^0\rangle & \langle B_0^1A_0^1\rangle & \langle B_0^1A_1^0\rangle & \langle B_0^1A_1^1\rangle & \langle B_0^1B_0^0\rangle & \langle B_0^1B_0^1\rangle & \langle B_0^1B_1^0\rangle & \langle B_0^1B_1^1\rangle \\
\langle B_1^0\rangle & \langle B_1^0A_0^0\rangle & \langle B_1^0A_0^1\rangle & \langle B_1^0A_1^0\rangle & \langle B_1^0A_1^1\rangle & \langle B_1^0B_0^0\rangle & \langle B_1^0B_0^1\rangle & \langle B_1^0B_1^0\rangle & \langle B_1^0B_1^1\rangle \\
\langle B_1^1\rangle & \langle B_1^1A_0^0\rangle & \langle B_1^1A_0^1\rangle & \langle B_1^1A_1^0\rangle & \langle B_1^1A_1^1\rangle & \langle B_1^1B_0^0\rangle & \langle B_1^1B_0^1\rangle & \langle B_1^1B_1^0\rangle & \langle B_1^1B_1^1\rangle
\end{bmatrix}
$$

Moment matrix

1. physical terms

$$\langle A_0^0\rangle = p^A(0|0),\ \langle B_0^1\rangle = p^B(0|1), \langle A_0^1 B_0^1\rangle = p(00|11),\ \langle A_0^1 B_1^1\rangle = p(01|11)$$

$$\Gamma_1 = \begin{bmatrix}
\langle\Psi|\Psi\rangle & \langle A_0^0\rangle & \langle A_0^1\rangle & \langle A_1^0\rangle & \langle A_1^1\rangle & \langle B_0^0\rangle & \langle B_0^1\rangle & \langle B_1^0\rangle & \langle B_1^1\rangle \\
\langle A_0^0\rangle & \langle A_0^0 A_0^0\rangle & \langle A_0^0 A_0^1\rangle & \langle A_0^0 A_1^0\rangle & \langle A_0^0 A_1^1\rangle & \langle A_0^0 B_0^0\rangle & \langle A_0^0 B_0^1\rangle & \langle A_0^0 B_1^0\rangle & \langle A_0^0 B_1^1\rangle \\
\langle A_0^1\rangle & \langle A_0^1 A_0^0\rangle & \langle A_0^1 A_0^1\rangle & \langle A_0^1 A_1^0\rangle & \langle A_0^1 A_1^1\rangle & \langle A_0^1 B_0^0\rangle & \langle A_0^1 B_0^1\rangle & \langle A_0^1 B_1^0\rangle & \langle A_0^1 B_1^1\rangle \\
\langle A_1^0\rangle & \langle A_1^0 A_0^0\rangle & \langle A_1^0 A_0^1\rangle & \langle A_1^0 A_1^0\rangle & \langle A_1^0 A_1^1\rangle & \langle A_1^0 B_0^0\rangle & \langle A_1^0 B_0^1\rangle & \langle A_1^0 B_1^0\rangle & \langle A_1^0 B_1^1\rangle \\
\langle A_1^1\rangle & \langle A_1^1 A_0^0\rangle & \langle A_1^1 A_0^1\rangle & \langle A_1^1 A_1^0\rangle & \langle A_1^1 A_1^1\rangle & \langle A_1^1 B_0^0\rangle & \langle A_1^1 B_0^1\rangle & \langle A_1^1 B_1^0\rangle & \langle A_1^1 B_1^1\rangle \\
\langle B_0^0\rangle & \langle B_0^0 A_0^0\rangle & \langle B_0^0 A_0^1\rangle & \langle B_0^0 A_1^0\rangle & \langle B_0^0 A_1^1\rangle & \langle B_0^0 B_0^0\rangle & \langle B_0^0 B_0^1\rangle & \langle B_0^0 B_1^0\rangle & \langle B_0^0 B_1^1\rangle \\
\langle B_0^1\rangle & \langle B_0^1 A_0^0\rangle & \langle B_0^1 A_0^1\rangle & \langle B_0^1 A_1^0\rangle & \langle B_0^1 A_1^1\rangle & \langle B_0^1 B_0^0\rangle & \langle B_0^1 B_0^1\rangle & \langle B_0^1 B_1^0\rangle & \langle B_0^1 B_1^1\rangle \\
\langle B_1^0\rangle & \langle B_1^0 A_0^0\rangle & \langle B_1^0 A_0^1\rangle & \langle B_1^0 A_1^0\rangle & \langle B_1^0 A_1^1\rangle & \langle B_1^0 B_0^0\rangle & \langle B_1^0 B_0^1\rangle & \langle B_1^0 B_1^0\rangle & \langle B_1^0 B_1^1\rangle \\
\langle B_1^1\rangle & \langle B_1^1 A_0^0\rangle & \langle B_1^1 A_0^1\rangle & \langle B_1^1 A_1^0\rangle & \langle B_1^1 A_1^1\rangle & \langle B_1^1 B_0^0\rangle & \langle B_1^1 B_0^1\rangle & \langle B_1^1 B_1^0\rangle & \langle B_1^1 B_1^1\rangle
\end{bmatrix}$$

Moment matrix

1. physical terms 2.normalization

$$\langle \Psi | \Psi \rangle = 1$$

$$\Gamma_1 = \begin{bmatrix}
\langle \Psi | \Psi \rangle & \langle A_0^0 \rangle & \langle A_0^1 \rangle & \langle A_1^0 \rangle & \langle A_1^1 \rangle & \langle B_0^0 \rangle & \langle B_0^1 \rangle & \langle B_1^0 \rangle & \langle B_1^1 \rangle \\
\langle A_0^0 \rangle & \langle A_0^0 A_0^0 \rangle & \langle A_0^0 A_0^1 \rangle & \langle A_0^0 A_1^0 \rangle & \langle A_0^0 A_1^1 \rangle & \langle A_0^0 B_0^0 \rangle & \langle A_0^0 B_0^1 \rangle & \langle A_0^0 B_1^0 \rangle & \langle A_0^0 B_1^1 \rangle \\
\langle A_0^1 \rangle & \langle A_0^1 A_0^0 \rangle & \langle A_0^1 A_0^1 \rangle & \langle A_0^1 A_1^0 \rangle & \langle A_0^1 A_1^1 \rangle & \langle A_0^1 B_0^0 \rangle & \langle A_0^1 B_0^1 \rangle & \langle A_0^1 B_1^0 \rangle & \langle A_0^1 B_1^1 \rangle \\
\langle A_1^0 \rangle & \langle A_1^0 A_0^0 \rangle & \langle A_1^0 A_0^1 \rangle & \langle A_1^0 A_1^0 \rangle & \langle A_1^0 A_1^1 \rangle & \langle A_1^0 B_0^0 \rangle & \langle A_1^0 B_0^1 \rangle & \langle A_1^0 B_1^0 \rangle & \langle A_1^0 B_1^1 \rangle \\
\langle A_1^1 \rangle & \langle A_1^1 A_0^0 \rangle & \langle A_1^1 A_0^1 \rangle & \langle A_1^1 A_1^0 \rangle & \langle A_1^1 A_1^1 \rangle & \langle A_1^1 B_0^0 \rangle & \langle A_1^1 B_0^1 \rangle & \langle A_1^1 B_1^0 \rangle & \langle A_1^1 B_1^1 \rangle \\
\langle B_0^0 \rangle & \langle B_0^0 A_0^0 \rangle & \langle B_0^0 A_0^1 \rangle & \langle B_0^0 A_1^0 \rangle & \langle B_0^0 A_1^1 \rangle & \langle B_0^0 B_0^0 \rangle & \langle B_0^0 B_0^1 \rangle & \langle B_0^0 B_1^0 \rangle & \langle B_0^0 B_1^1 \rangle \\
\langle B_0^1 \rangle & \langle B_0^1 A_0^0 \rangle & \langle B_0^1 A_0^1 \rangle & \langle B_0^1 A_1^0 \rangle & \langle B_0^1 A_1^1 \rangle & \langle B_0^1 B_0^0 \rangle & \langle B_0^1 B_0^1 \rangle & \langle B_0^1 B_1^0 \rangle & \langle B_0^1 B_1^1 \rangle \\
\langle B_1^0 \rangle & \langle B_1^0 A_0^0 \rangle & \langle B_1^0 A_0^1 \rangle & \langle B_1^0 A_1^0 \rangle & \langle B_1^0 A_1^1 \rangle & \langle B_1^0 B_0^0 \rangle & \langle B_1^0 B_0^1 \rangle & \langle B_1^0 B_1^0 \rangle & \langle B_1^0 B_1^1 \rangle \\
\langle B_1^1 \rangle & \langle B_1^1 A_0^0 \rangle & \langle B_1^1 A_0^1 \rangle & \langle B_1^1 A_1^0 \rangle & \langle B_1^1 A_1^1 \rangle & \langle B_1^1 B_0^0 \rangle & \langle B_1^1 B_0^1 \rangle & \langle B_1^1 B_1^0 \rangle & \langle B_1^1 B_1^1 \rangle
\end{bmatrix}$$

Moment matrix

1. physical terms 2.normalization 3. completeness

$$\sum_a A_a^x = \mathbb{1}$$

$$\Gamma_1 = \begin{bmatrix}
\langle\Psi|\Psi\rangle & \langle A_0^0\rangle & \langle A_0^1\rangle & \langle A_1^0\rangle & \langle A_1^1\rangle & \langle B_0^0\rangle & \langle B_0^1\rangle & \langle B_1^0\rangle & \langle B_1^1\rangle \\
\langle A_0^0\rangle & \langle A_0^0A_0^0\rangle & \langle A_0^0A_0^1\rangle & \langle A_0^0A_1^0\rangle & \langle A_0^0A_1^1\rangle & \langle A_0^0B_0^0\rangle & \langle A_0^0B_0^1\rangle & \langle A_0^0B_1^0\rangle & \langle A_0^0B_1^1\rangle \\
\langle A_0^1\rangle & \langle A_0^1A_0^0\rangle & \langle A_0^1A_0^1\rangle & \langle A_0^1A_1^0\rangle & \langle A_0^1A_1^1\rangle & \langle A_0^1B_0^0\rangle & \langle A_0^1B_0^1\rangle & \langle A_0^1B_1^0\rangle & \langle A_0^1B_1^1\rangle \\
\langle A_1^0\rangle & \langle A_1^0A_0^0\rangle & \langle A_1^0A_0^1\rangle & \langle A_1^0A_1^0\rangle & \langle A_1^0A_1^1\rangle & \langle A_1^0B_0^0\rangle & \langle A_1^0B_0^1\rangle & \langle A_1^0B_1^0\rangle & \langle A_1^0B_1^1\rangle \\
\langle A_1^1\rangle & \langle A_1^1A_0^0\rangle & \langle A_1^1A_0^1\rangle & \langle A_1^1A_1^0\rangle & \langle A_1^1A_1^1\rangle & \langle A_1^1B_0^0\rangle & \langle A_1^1B_0^1\rangle & \langle A_1^1B_1^0\rangle & \langle A_1^1B_1^1\rangle \\
\langle B_0^0\rangle & \langle B_0^0A_0^0\rangle & \langle B_0^0A_0^1\rangle & \langle B_0^0A_1^0\rangle & \langle B_0^0A_1^1\rangle & \langle B_0^0B_0^0\rangle & \langle B_0^0B_0^1\rangle & \langle B_0^0B_1^0\rangle & \langle B_0^0B_1^1\rangle \\
\langle B_0^1\rangle & \langle B_0^1A_0^0\rangle & \langle B_0^1A_0^1\rangle & \langle B_0^1A_1^0\rangle & \langle B_0^1A_1^1\rangle & \langle B_0^1B_0^0\rangle & \langle B_0^1B_0^1\rangle & \langle B_0^1B_1^0\rangle & \langle B_0^1B_1^1\rangle \\
\langle B_1^0\rangle & \langle B_1^0A_0^0\rangle & \langle B_1^0A_0^1\rangle & \langle B_1^0A_1^0\rangle & \langle B_1^0A_1^1\rangle & \langle B_1^0B_0^0\rangle & \langle B_1^0B_0^1\rangle & \langle B_1^0B_1^0\rangle & \langle B_1^0B_1^1\rangle \\
\langle B_1^1\rangle & \langle B_1^1A_0^0\rangle & \langle B_1^1A_0^1\rangle & \langle B_1^1A_1^0\rangle & \langle B_1^1A_1^1\rangle & \langle B_1^1B_0^0\rangle & \langle B_1^1B_0^1\rangle & \langle B_1^1B_1^0\rangle & \langle B_1^1B_1^1\rangle
\end{bmatrix}$$

Moment matrix

1. physical terms 2.normalization 3. completeness 4. orthogonality

$$\langle A^x_a A^x_{a'} \rangle = 0 \quad \forall\, a \neq a'$$

$$\Gamma_1 = \begin{bmatrix}
\langle \Psi|\Psi\rangle & \langle A^0_0\rangle & \langle A^1_0\rangle & \langle A^0_1\rangle & \langle A^1_1\rangle & \langle B^0_0\rangle & \langle B^1_0\rangle & \langle B^0_1\rangle & \langle B^1_1\rangle \\
\langle A^0_0\rangle & \langle A^0_0A^0_0\rangle & \langle A^0_0A^1_0\rangle & \langle A^0_0A^0_1\rangle & \langle A^0_0A^1_1\rangle & \langle A^0_0B^0_0\rangle & \langle A^0_0B^1_0\rangle & \langle A^0_0B^0_1\rangle & \langle A^0_0B^1_1\rangle \\
\langle A^1_0\rangle & \langle A^1_0A^0_0\rangle & \langle A^1_0A^1_0\rangle & \langle A^1_0A^0_1\rangle & \langle A^1_0A^1_1\rangle & \langle A^1_0B^0_0\rangle & \langle A^1_0B^1_0\rangle & \langle A^1_0B^0_1\rangle & \langle A^1_0B^1_1\rangle \\
\langle A^0_1\rangle & \langle A^0_1A^0_0\rangle & \langle A^0_1A^1_0\rangle & \langle A^0_1A^0_1\rangle & \langle A^0_1A^1_1\rangle & \langle A^0_1B^0_0\rangle & \langle A^0_1B^1_0\rangle & \langle A^0_1B^0_1\rangle & \langle A^0_1B^1_1\rangle \\
\langle A^1_1\rangle & \langle A^1_1A^0_0\rangle & \langle A^1_1A^1_0\rangle & \langle A^1_1A^0_1\rangle & \langle A^1_1A^1_1\rangle & \langle A^1_1B^0_0\rangle & \langle A^1_1B^1_0\rangle & \langle A^1_1B^0_1\rangle & \langle A^1_1B^1_1\rangle \\
\langle B^0_0\rangle & \langle B^0_0A^0_0\rangle & \langle B^0_0A^1_0\rangle & \langle B^0_0A^0_1\rangle & \langle B^0_0A^1_1\rangle & \langle B^0_0B^0_0\rangle & \langle B^0_0B^1_0\rangle & \langle B^0_0B^0_1\rangle & \langle B^0_0B^1_1\rangle \\
\langle B^1_0\rangle & \langle B^1_0A^0_0\rangle & \langle B^1_0A^1_0\rangle & \langle B^1_0A^0_1\rangle & \langle B^1_0A^1_1\rangle & \langle B^1_0B^0_0\rangle & \langle B^1_0B^1_0\rangle & \langle B^1_0B^0_1\rangle & \langle B^1_0B^1_1\rangle \\
\langle B^0_1\rangle & \langle B^0_1A^0_0\rangle & \langle B^0_1A^1_0\rangle & \langle B^0_1A^0_1\rangle & \langle B^0_1A^1_1\rangle & \langle B^0_1B^0_0\rangle & \langle B^0_1B^1_0\rangle & \langle B^0_1B^0_1\rangle & \langle B^0_1B^1_1\rangle \\
\langle B^1_1\rangle & \langle B^1_1A^0_0\rangle & \langle B^1_1A^1_0\rangle & \langle B^1_1A^0_1\rangle & \langle B^1_1A^1_1\rangle & \langle B^1_1B^0_0\rangle & \langle B^1_1B^1_0\rangle & \langle B^1_1B^0_1\rangle & \langle B^1_1B^1_1\rangle
\end{bmatrix}$$

Moment matrix

1. physical terms 2.normalization 3. completeness 4. orthogonality 5. projectivity

$$A_a^x A_a^x = A_a^x$$

$$\Gamma_1 = \begin{bmatrix}
\langle \Psi|\Psi \rangle & \langle A_0^0 \rangle & \langle A_0^1 \rangle & \langle A_1^0 \rangle & \langle A_1^1 \rangle & \langle B_0^0 \rangle & \langle B_0^1 \rangle & \langle B_1^0 \rangle & \langle B_1^1 \rangle \\
\langle A_0^0 \rangle & \langle A_0^0 A_0^0 \rangle & \langle A_0^0 A_0^1 \rangle & \langle A_0^0 A_1^0 \rangle & \langle A_0^0 A_1^1 \rangle & \langle A_0^0 B_0^0 \rangle & \langle A_0^0 B_0^1 \rangle & \langle A_0^0 B_1^0 \rangle & \langle A_0^0 B_1^1 \rangle \\
\langle A_0^1 \rangle & \langle A_0^1 A_0^0 \rangle & \langle A_0^1 A_0^1 \rangle & \langle A_0^1 A_1^0 \rangle & \langle A_0^1 A_1^1 \rangle & \langle A_0^1 B_0^0 \rangle & \langle A_0^1 B_0^1 \rangle & \langle A_0^1 B_1^0 \rangle & \langle A_0^1 B_1^1 \rangle \\
\langle A_1^0 \rangle & \langle A_1^0 A_0^0 \rangle & \langle A_1^0 A_0^1 \rangle & \langle A_1^0 A_1^0 \rangle & \langle A_1^0 A_1^1 \rangle & \langle A_1^0 B_0^0 \rangle & \langle A_1^0 B_0^1 \rangle & \langle A_1^0 B_1^0 \rangle & \langle A_1^0 B_1^1 \rangle \\
\langle A_1^1 \rangle & \langle A_1^1 A_0^0 \rangle & \langle A_1^1 A_0^1 \rangle & \langle A_1^1 A_1^0 \rangle & \langle A_1^1 A_1^1 \rangle & \langle A_1^1 B_0^0 \rangle & \langle A_1^1 B_0^1 \rangle & \langle A_1^1 B_1^0 \rangle & \langle A_1^1 B_1^1 \rangle \\
\langle B_0^0 \rangle & \langle B_0^0 A_0^0 \rangle & \langle B_0^0 A_0^1 \rangle & \langle B_0^0 A_1^0 \rangle & \langle B_0^0 A_1^1 \rangle & \langle B_0^0 B_0^0 \rangle & \langle B_0^0 B_0^1 \rangle & \langle B_0^0 B_1^0 \rangle & \langle B_0^0 B_1^1 \rangle \\
\langle B_0^1 \rangle & \langle B_0^1 A_0^0 \rangle & \langle B_0^1 A_0^1 \rangle & \langle B_0^1 A_1^0 \rangle & \langle B_0^1 A_1^1 \rangle & \langle B_0^1 B_0^0 \rangle & \langle B_0^1 B_0^1 \rangle & \langle B_0^1 B_1^0 \rangle & \langle B_0^1 B_1^1 \rangle \\
\langle B_1^0 \rangle & \langle B_1^0 A_0^0 \rangle & \langle B_1^0 A_0^1 \rangle & \langle B_1^0 A_1^0 \rangle & \langle B_1^0 A_1^1 \rangle & \langle B_1^0 B_0^0 \rangle & \langle B_1^0 B_0^1 \rangle & \langle B_1^0 B_1^0 \rangle & \langle B_1^0 B_1^1 \rangle \\
\langle B_1^1 \rangle & \langle B_1^1 A_0^0 \rangle & \langle B_1^1 A_0^1 \rangle & \langle B_1^1 A_1^0 \rangle & \langle B_1^1 A_1^1 \rangle & \langle B_1^1 B_0^0 \rangle & \langle B_1^1 B_0^1 \rangle & \langle B_1^1 B_1^0 \rangle & \langle B_1^1 B_1^1 \rangle
\end{bmatrix}$$

Moment matrix

1. physical terms 2.normalization 3. completeness 4. orthogonality 5. projectivity 6. commutativity

$$B_1^0 A_0^0 = A_0^0 B_1^0$$

$$\Gamma_1 = \begin{bmatrix}
\langle\Psi|\Psi\rangle & \langle A_0^0\rangle & \langle A_0^1\rangle & \langle A_1^0\rangle & \langle A_1^1\rangle & \langle B_0^0\rangle & \langle B_0^1\rangle & \langle B_1^0\rangle & \langle B_1^1\rangle \\
\langle A_0^0\rangle & \langle A_0^0 A_0^0\rangle & \langle A_0^0 A_0^1\rangle & \langle A_0^0 A_1^0\rangle & \langle A_0^0 A_1^1\rangle & \langle A_0^0 B_0^0\rangle & \langle A_0^0 B_0^1\rangle & \langle A_0^0 B_1^0\rangle & \langle A_0^0 B_1^1\rangle \\
\langle A_0^1\rangle & \langle A_0^1 A_0^0\rangle & \langle A_0^1 A_0^1\rangle & \langle A_0^1 A_1^0\rangle & \langle A_0^1 A_1^1\rangle & \langle A_0^1 B_0^0\rangle & \langle A_0^1 B_0^1\rangle & \langle A_0^1 B_1^0\rangle & \langle A_0^1 B_1^1\rangle \\
\langle A_1^0\rangle & \langle A_1^0 A_0^0\rangle & \langle A_1^0 A_0^1\rangle & \langle A_1^0 A_1^0\rangle & \langle A_1^0 A_1^1\rangle & \langle A_1^0 B_0^0\rangle & \langle A_1^0 B_0^1\rangle & \langle A_1^0 B_1^0\rangle & \langle A_1^0 B_1^1\rangle \\
\langle A_1^1\rangle & \langle A_1^1 A_0^0\rangle & \langle A_1^1 A_0^1\rangle & \langle A_1^1 A_1^0\rangle & \langle A_1^1 A_1^1\rangle & \langle A_1^1 B_0^0\rangle & \langle A_1^1 B_0^1\rangle & \langle A_1^1 B_1^0\rangle & \langle A_1^1 B_1^1\rangle \\
\langle B_0^0\rangle & \langle B_0^0 A_0^0\rangle & \langle B_0^0 A_0^1\rangle & \langle B_0^0 A_1^0\rangle & \langle B_0^0 A_1^1\rangle & \langle B_0^0 B_0^0\rangle & \langle B_0^0 B_0^1\rangle & \langle B_0^0 B_1^0\rangle & \langle B_0^0 B_1^1\rangle \\
\langle B_0^1\rangle & \langle B_0^1 A_0^0\rangle & \langle B_0^1 A_0^1\rangle & \langle B_0^1 A_1^0\rangle & \langle B_0^1 A_1^1\rangle & \langle B_0^1 B_0^0\rangle & \langle B_0^1 B_0^1\rangle & \langle B_0^1 B_1^0\rangle & \langle B_0^1 B_1^1\rangle \\
\langle B_1^0\rangle & \langle B_1^0 A_0^0\rangle & \langle B_1^0 A_0^1\rangle & \langle B_1^0 A_1^0\rangle & \langle B_1^0 A_1^1\rangle & \langle B_1^0 B_0^0\rangle & \langle B_1^0 B_0^1\rangle & \langle B_1^0 B_1^0\rangle & \langle B_1^0 B_1^1\rangle \\
\langle B_1^1\rangle & \langle B_1^1 A_0^0\rangle & \langle B_1^1 A_0^1\rangle & \langle B_1^1 A_1^0\rangle & \langle B_1^1 A_1^1\rangle & \langle B_1^1 B_0^0\rangle & \langle B_1^1 B_0^1\rangle & \langle B_1^1 B_1^0\rangle & \langle B_1^1 B_1^1\rangle
\end{bmatrix}$$

Moment matrix

1. physical terms 2.normalization 3. completeness 4. orthogonality 5. projectivity 6. commutativity 7. PSD

$$\Gamma_1 \geq 0$$

$$\Gamma_1 = \begin{bmatrix}
\langle \Psi | \Psi \rangle & \langle A_0^0 \rangle & \langle A_0^1 \rangle & \langle A_1^0 \rangle & \langle A_1^1 \rangle & \langle B_0^0 \rangle & \langle B_0^1 \rangle & \langle B_1^0 \rangle & \langle B_1^1 \rangle \\
\langle A_0^0 \rangle & \langle A_0^0 A_0^0 \rangle & \langle A_0^0 A_0^1 \rangle & \langle A_0^0 A_1^0 \rangle & \langle A_0^0 A_1^1 \rangle & \langle A_0^0 B_0^0 \rangle & \langle A_0^0 B_0^1 \rangle & \langle A_0^0 B_1^0 \rangle & \langle A_0^0 B_1^1 \rangle \\
\langle A_0^1 \rangle & \langle A_0^1 A_0^0 \rangle & \langle A_0^1 A_0^1 \rangle & \langle A_0^1 A_1^0 \rangle & \langle A_0^1 A_1^1 \rangle & \langle A_0^1 B_0^0 \rangle & \langle A_0^1 B_0^1 \rangle & \langle A_0^1 B_1^0 \rangle & \langle A_0^1 B_1^1 \rangle \\
\langle A_1^0 \rangle & \langle A_1^0 A_0^0 \rangle & \langle A_1^0 A_0^1 \rangle & \langle A_1^0 A_1^0 \rangle & \langle A_1^0 A_1^1 \rangle & \langle A_1^0 B_0^0 \rangle & \langle A_1^0 B_0^1 \rangle & \langle A_1^0 B_1^0 \rangle & \langle A_1^0 B_1^1 \rangle \\
\langle A_1^1 \rangle & \langle A_1^1 A_0^0 \rangle & \langle A_1^1 A_0^1 \rangle & \langle A_1^1 A_1^0 \rangle & \langle A_1^1 A_1^1 \rangle & \langle A_1^1 B_0^0 \rangle & \langle A_1^1 B_0^1 \rangle & \langle A_1^1 B_1^0 \rangle & \langle A_1^1 B_1^1 \rangle \\
\langle B_0^0 \rangle & \langle B_0^0 A_0^0 \rangle & \langle B_0^0 A_0^1 \rangle & \langle B_0^0 A_1^0 \rangle & \langle B_0^0 A_1^1 \rangle & \langle B_0^0 B_0^0 \rangle & \langle B_0^0 B_0^1 \rangle & \langle B_0^0 B_1^0 \rangle & \langle B_0^0 B_1^1 \rangle \\
\langle B_0^1 \rangle & \langle B_0^1 A_0^0 \rangle & \langle B_0^1 A_0^1 \rangle & \langle B_0^1 A_1^0 \rangle & \langle B_0^1 A_1^1 \rangle & \langle B_0^1 B_0^0 \rangle & \langle B_0^1 B_0^1 \rangle & \langle B_0^1 B_1^0 \rangle & \langle B_0^1 B_1^1 \rangle \\
\langle B_1^0 \rangle & \langle B_1^0 A_0^0 \rangle & \langle B_1^0 A_0^1 \rangle & \langle B_1^0 A_1^0 \rangle & \langle B_1^0 A_1^1 \rangle & \langle B_1^0 B_0^0 \rangle & \langle B_1^0 B_0^1 \rangle & \langle B_1^0 B_1^0 \rangle & \langle B_1^0 B_1^1 \rangle \\
\langle B_1^1 \rangle & \langle B_1^1 A_0^0 \rangle & \langle B_1^1 A_0^1 \rangle & \langle B_1^1 A_1^0 \rangle & \langle B_1^1 A_1^1 \rangle & \langle B_1^1 B_0^0 \rangle & \langle B_1^1 B_0^1 \rangle & \langle B_1^1 B_1^0 \rangle & \langle B_1^1 B_1^1 \rangle
\end{bmatrix}$$

we can now go back to problem

$$
\begin{aligned}
given \quad & p(ab|xy) \\
find \quad & \Psi,\ \{A_a^x\},\ \{B_b^y\} \\
s.t. \quad & p(ab|xy) = Tr[\Psi A_a^x B_b^y] \quad \forall a, b, x, y \\
& \Psi \text{ is a state} \\
& \{A_a^x\},\ \{B_b^y\} \text{ are measurements}
\end{aligned}
$$

hard

$$
\begin{aligned}
given \quad & p(ab|xy) \\
find \quad & \Gamma_1
\end{aligned}
$$

$s.t.$

1. known terms
2. normalization
3. completeness
4. orthogonality
5. projectivity
6. commutativity
7. PSD

doable!

we can now go back to problem

$$\begin{aligned} &given \quad p(ab|xy) \\ &find \quad \Psi, \{A_a^x\}, \{B_b^y\} \\ &s.t. \quad p(ab|xy) = Tr[\Psi A_a^x B_b^y] \quad \forall a, b, x, y \\ &\qquad \Psi \text{ is a state} \\ &\qquad \{A_a^x\}, \{B_b^y\} \text{ are measurements} \end{aligned}$$

$$\begin{aligned} &given \quad p(ab|xy) \\ &find \quad \Gamma_1 \end{aligned}$$

s.t.
1. known terms
2. normalization
3. completeness
4. orthogonality
5. projectivity
6. commutativity
7. PSD

feasible $\longrightarrow$ feasible

we can now go back to problem

$$\begin{aligned} given \quad & p(ab|xy) \\ find \quad & \Psi,\ \{A_a^x\},\ \{B_b^y\} \\ s.t. \quad & p(ab|xy) = Tr[\Psi A_a^x B_b^y] \quad \forall a,b,x,y \\ & \Psi \text{ is a state} \\ & \{A_a^x\},\ \{B_b^y\} \text{ are measurements} \end{aligned}$$

$$\begin{aligned} given \quad & p(ab|xy) \\ find \quad & \Gamma_1 \\ s.t. \quad & \text{1. known terms} \\ & \text{2. normalization} \\ & \text{3. completeness} \\ & \text{4. orthogonality} \\ & \text{5. projectivity} \\ & \text{6. commutativity} \\ & \text{7. PSD} \end{aligned}$$

<u>in</u>feasible ← <u>in</u>feasible

we can now go back to problem

$$
\begin{aligned}
given \quad & p(ab|xy) \\
find \quad & \Psi,\ \{A_a^x\},\ \{B_b^y\} \\
s.t. \quad & p(ab|xy) = Tr[\Psi A_a^x B_b^y] \quad \forall a, b, x, y \\
& \Psi \text{ is a state} \\
& \{A_a^x\},\ \{B_b^y\} \text{ are measurements}
\end{aligned}
$$

$$
\begin{aligned}
given \quad & p(ab|xy) \\
find \quad & \Gamma_1 \\
s.t. \quad & 1.\ \text{known terms} \\
& 2.\ \text{normalization} \\
& 3.\ \text{completeness} \\
& 4.\ \text{orthogonality} \\
& 5.\ \text{projectivity} \\
& 6.\ \text{commutativity} \\
& 7.\ \text{PSD}
\end{aligned}
$$

?????? $\longleftarrow$ feasible

Consider a higher level of hierarchy

$$S_1 = \{ \mathbb{1} \} \cup \{A_a^x\} \cup \{B_b^y\}$$

$$S_2 = S_1 \cup \{A_a^x A_{a'}^{x'}\} \cup \{A_a^x B_b^y\} \cup \{B_b^y B_{b'}^{y'}\}$$

Construct higher moment matrix

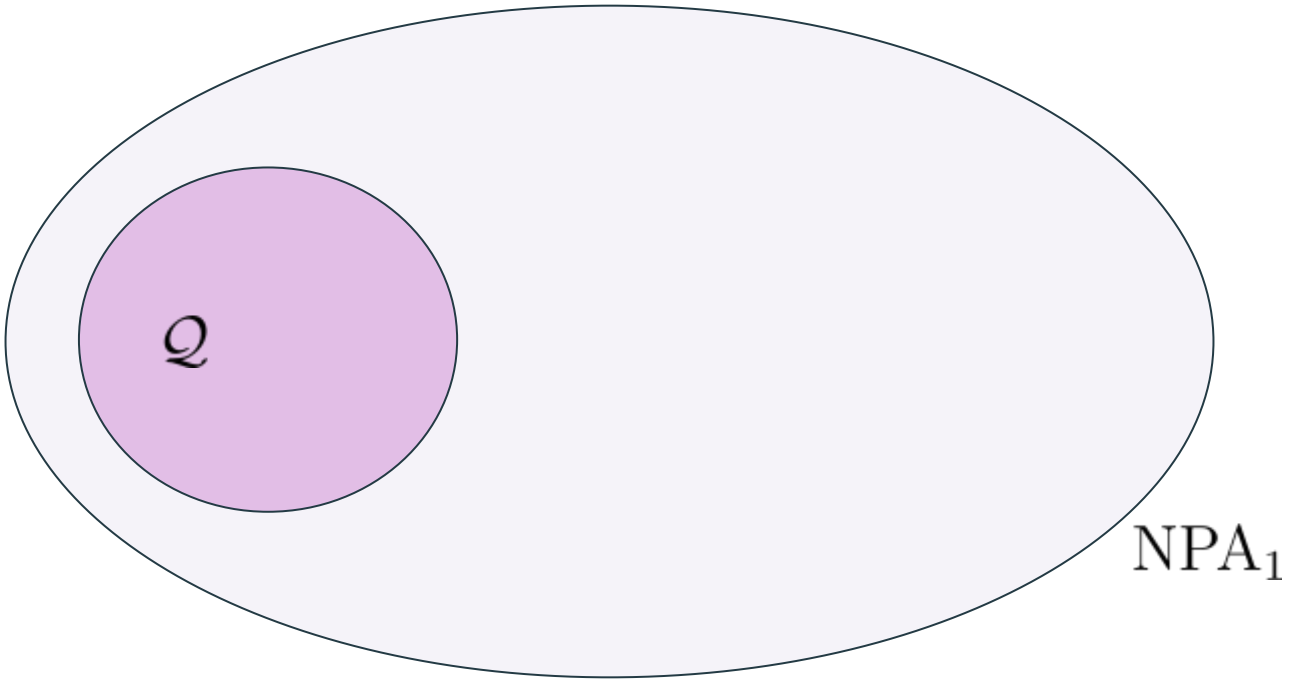
$$\Gamma_{ij} = \langle \Psi | O_i^\dagger O_j | \Psi \rangle, \quad O_i, O_j \in S_2$$

new elements, e.g.

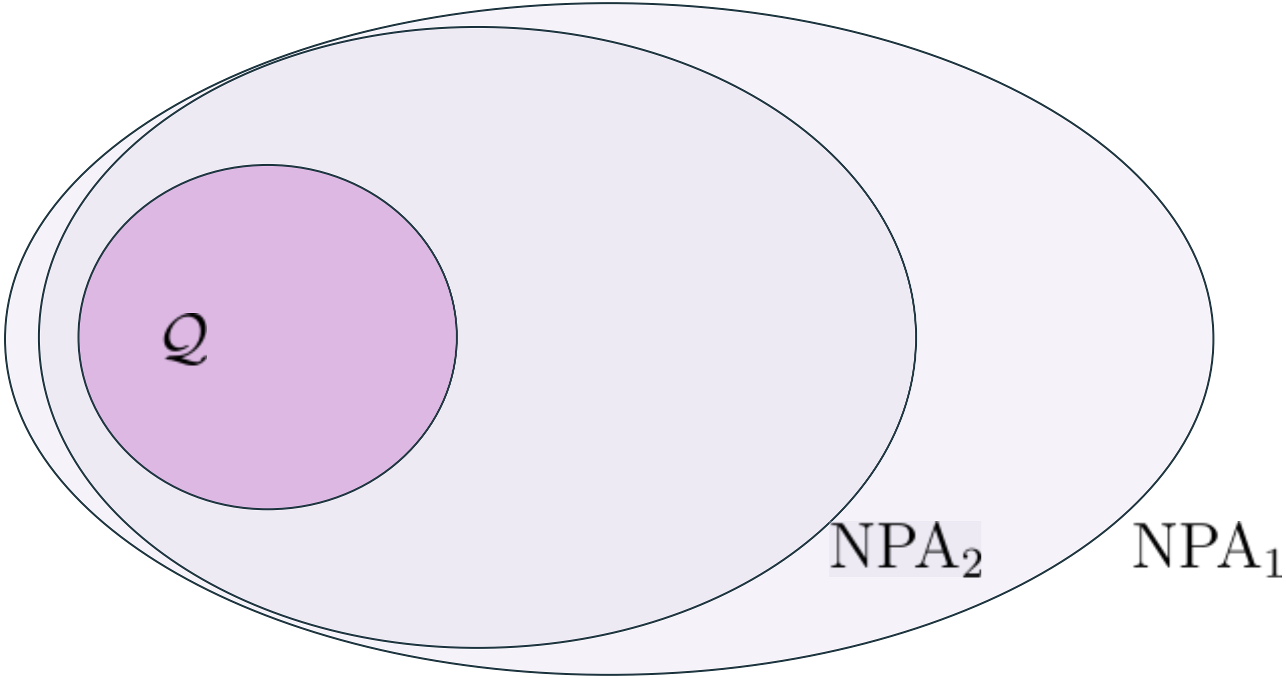
$$\langle A_0^0 A_0^1 B_0^1 \rangle, \ \langle A_1^0 A_1^1 B_0^0 B_1^1 \rangle, \ \langle B_0^0 B_0^1 B_1^1 \rangle$$

$$\begin{array}{ll} \textit{given} & p(ab|xy) \\ \textit{find} & \Gamma_2 \\ & \\ s.t. & \text{1. known terms} \\ & \text{2. normalization} \\ & \text{3. completeness} \\ & \text{4. orthogonality} \\ & \text{5. projectivity} \\ & \text{6. commutativity} \\ & \text{7. PSD} \end{array}$$

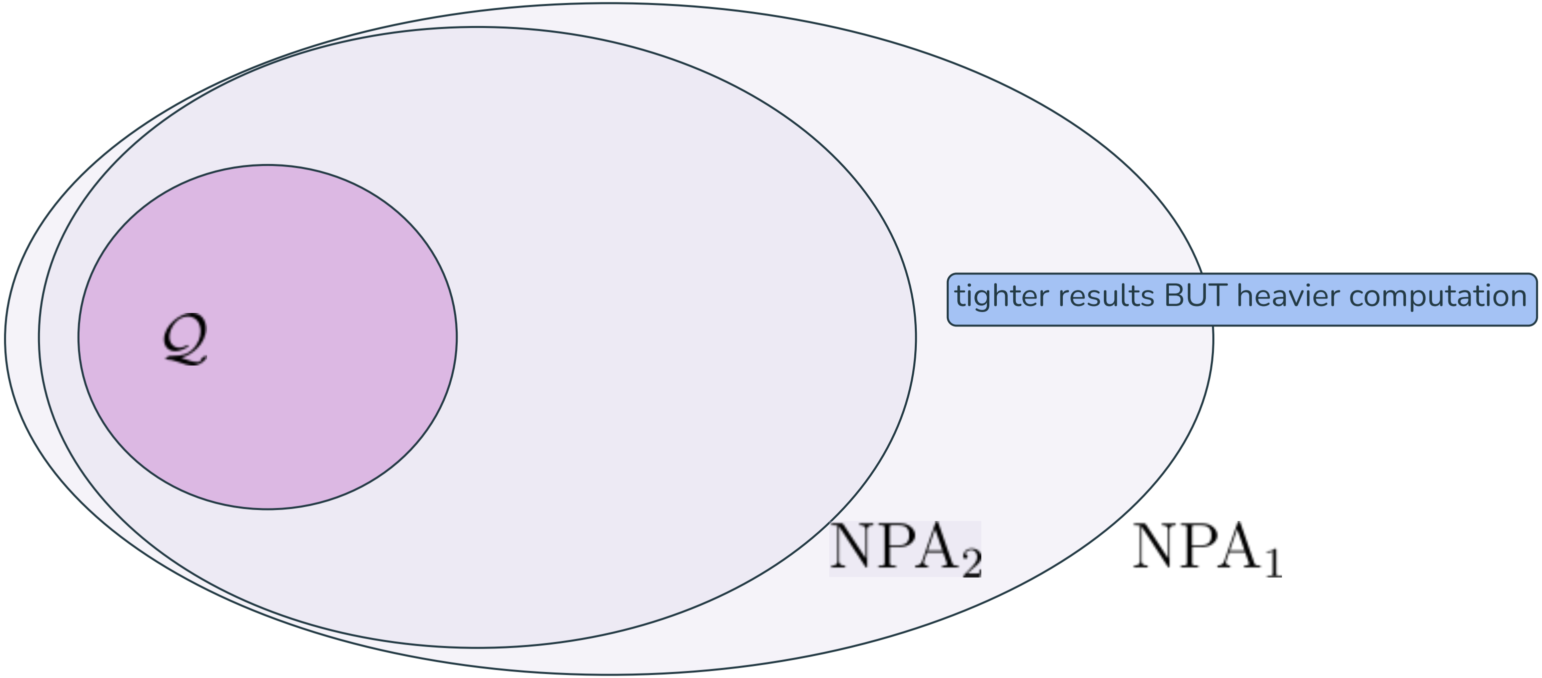
An infinite hierarchy



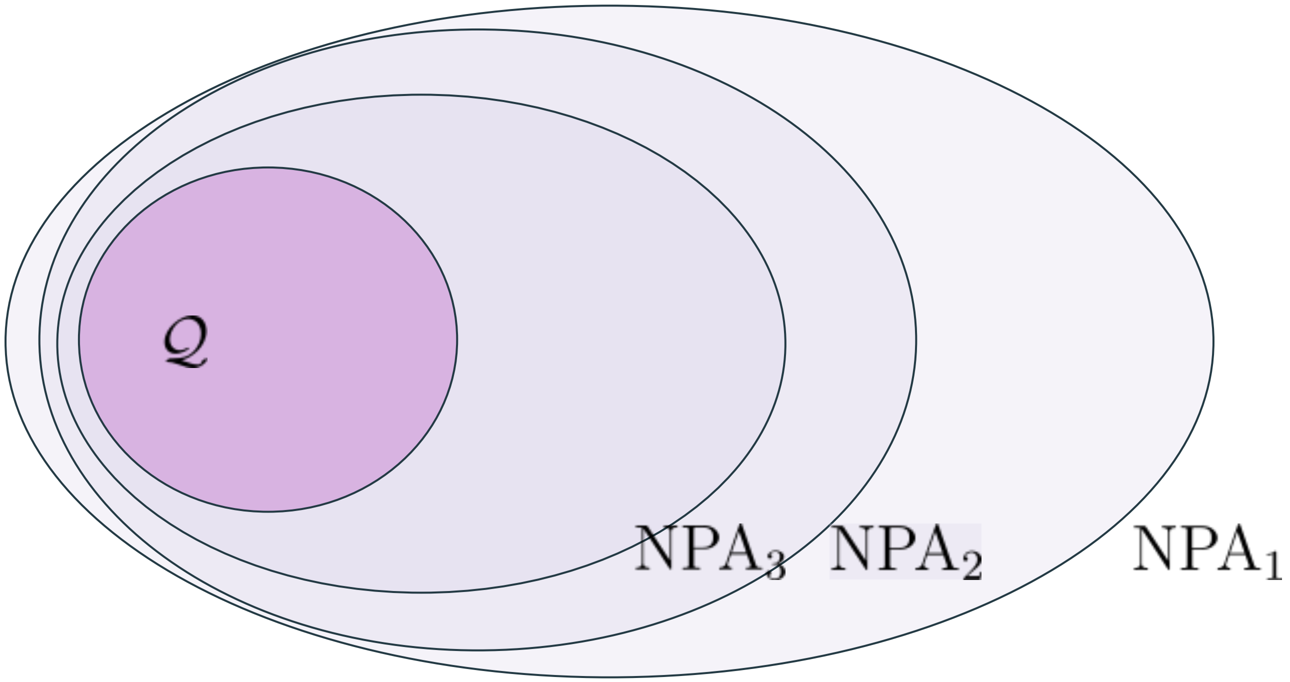
An infinite hierarchy



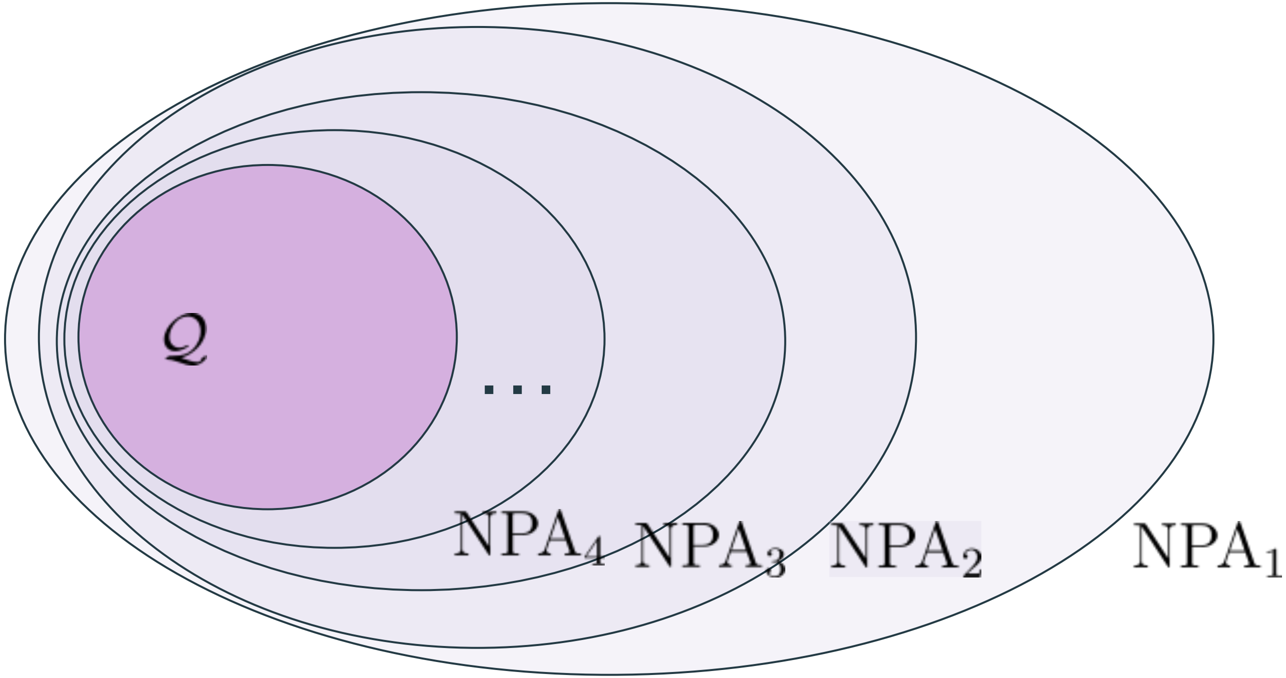
An infinite hierarchy



An infinite hierarchy

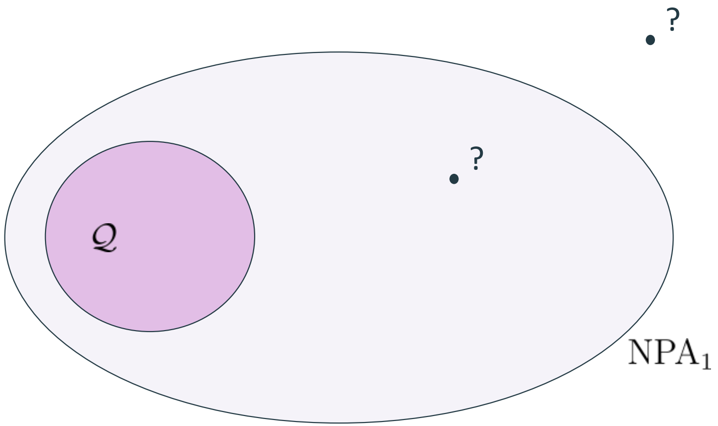


An infinite hierarchy



The dual approach: SOS

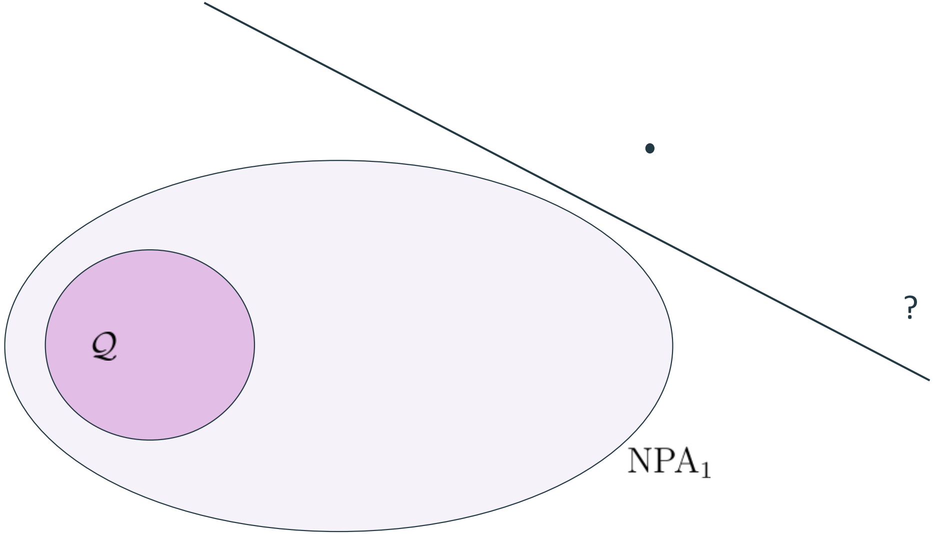
$$
\begin{array}{ll}
\textit{given} & p(ab|xy) \\
\textit{find} & \Gamma_2 \\
 & \\
s.t. & 1.\ \text{known terms} \\
 & 2.\ \text{normalization} \\
 & 3.\ \text{completeness} \\
 & 4.\ \text{orthogonality} \\
 & 5.\ \text{projectivity} \\
 & 6.\ \text{commutativity} \\
 & 7.\ \text{PSD}
\end{array}
$$



The dual approach: SOS

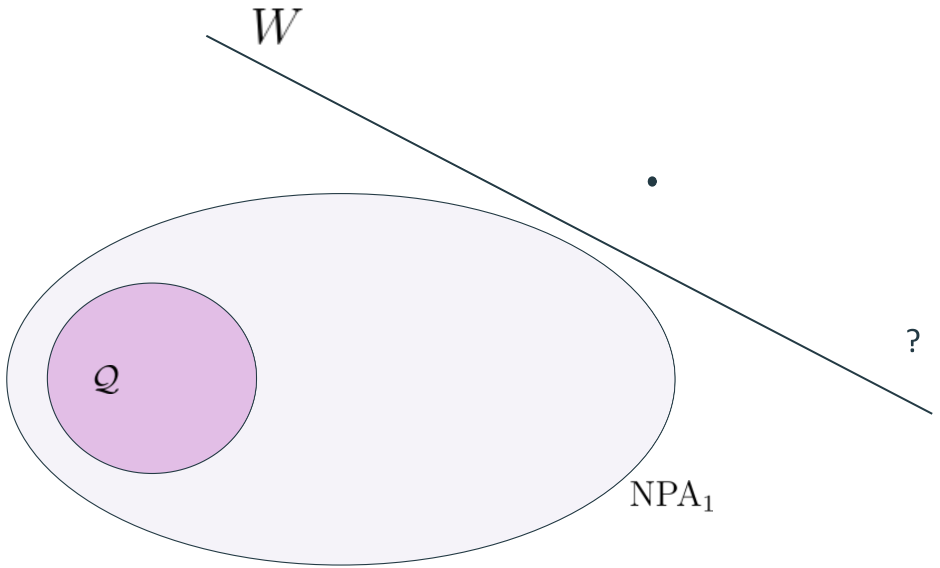
given $p(ab|xy)$
find Γ_2

s.t.
1. known terms
2. normalization
3. completeness
4. orthogonality
5. projectivity
6. commutativity
7. PSD

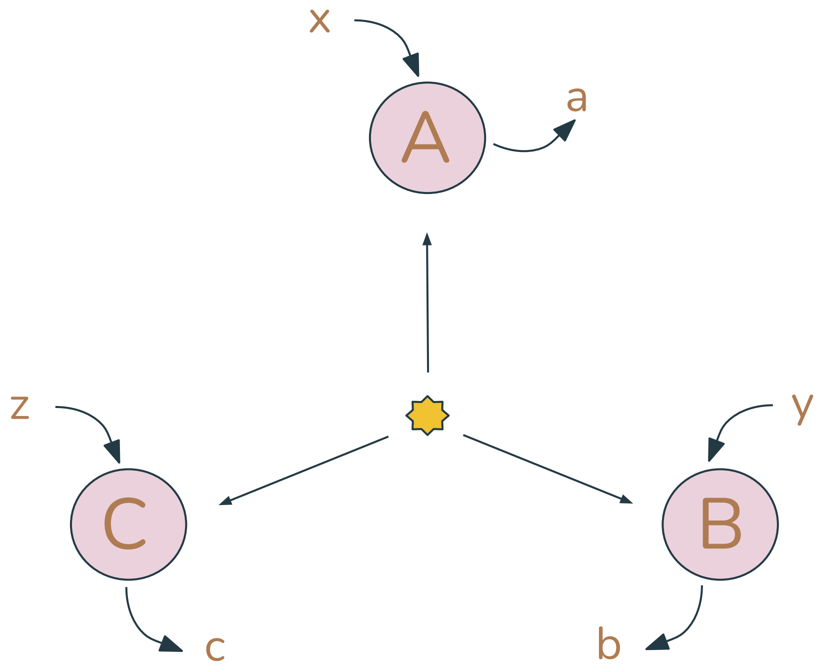


The dual approach: SOS

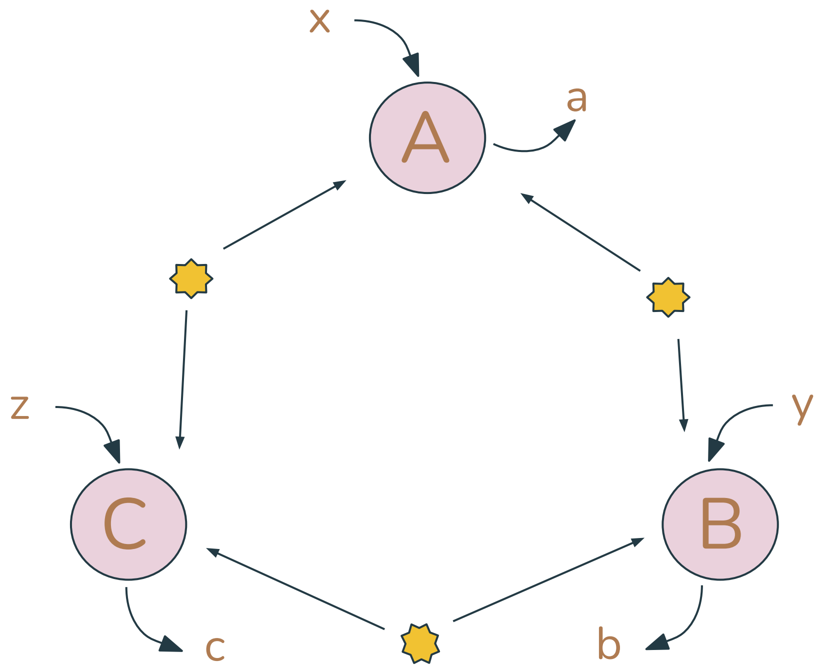
$$\begin{array}{ll} \textit{given} & p(ab|xy) \\ \textit{find} & \Gamma_2 \\ & \\ s.t. & \begin{array}{l} 1.\ \text{known terms} \\ 2.\ \text{normalization} \\ 3.\ \text{completeness} \\ 4.\ \text{orthogonality} \\ 5.\ \text{projectivity} \\ 6.\ \text{commutativity} \\ 7.\ \text{PSD} \end{array} \end{array}$$



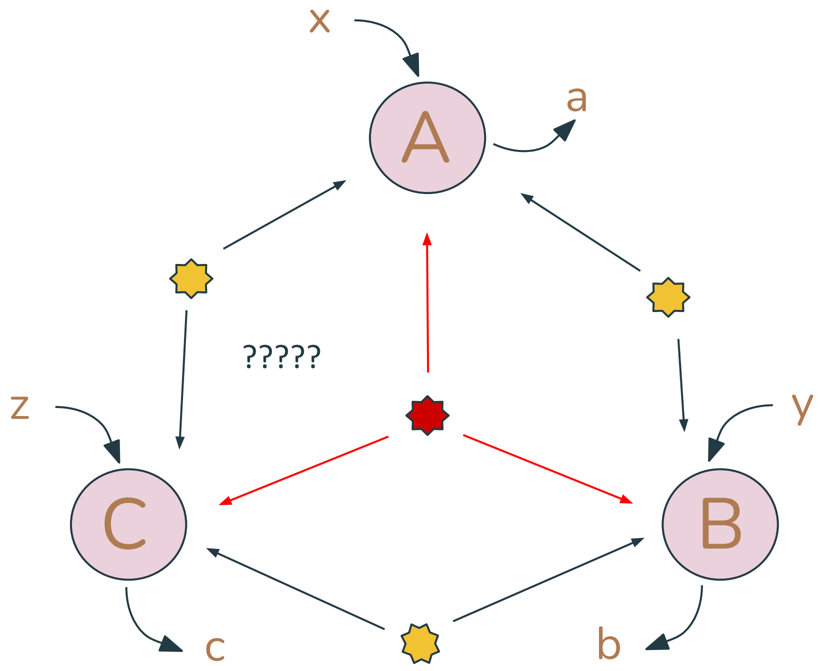
for what is it NOT enough?



for what is it NOT enough?



for what is it NOT enough?



for what is it NOT enough?