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Round Elimination

Meta-algorithmics

- **Normal algorithms** — example:
  - input: graph G
  - output: coloring of graph G
- **Meta-algorithms** — example:
  - input: ***computational problem*** P
  - output: ***algorithm*** for solving P

How to represent problems or algorithms?

Round elimination

- Basic idea already used by Linial (1987)
  - *"it is not possible to 3-color cycles in* $o(\log^* n)$ *rounds"*
- Until 2015 it was thought this is an ad-hoc trick that only works for graph coloring
- **Lots** of new applications since 2016
- General idea formalized in 2019

Proving lower bounds

- **Claim:** solving problem X ***takes ≥ 5 rounds***
- **Equivalent:** ***any 4-round algorithm A fails*** to solve problem X
- How to show something like this?
  - huge number of possible 4-round algorithms

Proving lower bounds

- **Easy to do directly:**
  showing that 0-round algorithms fail
- **Hard to do directly:**
  showing that 4-round algorithms fail
- Solution: ***round elimination technique***

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

→ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

→ A_2 solves problem $X_2 = \mathrm{re}(X_1)$ in 2 rounds

→ A_3 solves problem $X_3 = \mathrm{re}(X_2)$ in 1 round

→ A_4 solves problem $X_4 = \mathrm{re}(X_3)$ in 0 rounds

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

→ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

→ A_2 solves problem $X_2 = \mathrm{re}(X_1)$ in 2 rounds

→ A_3 solves problem $X_3 = \mathrm{re}(X_2)$ in 1 round

→ A_4 solves problem $X_4 = \mathrm{re}(X_3)$ in 0 rounds

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

→ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

→ A_2 solves problem $X_2 = \mathrm{re}(X_1)$ in 2 rounds

→ A_3 solves problem $X_3 = \mathrm{re}(X_2)$ in 1 round

→ A_4 solves problem $X_4 = \mathrm{re}(X_3)$ in 0 rounds

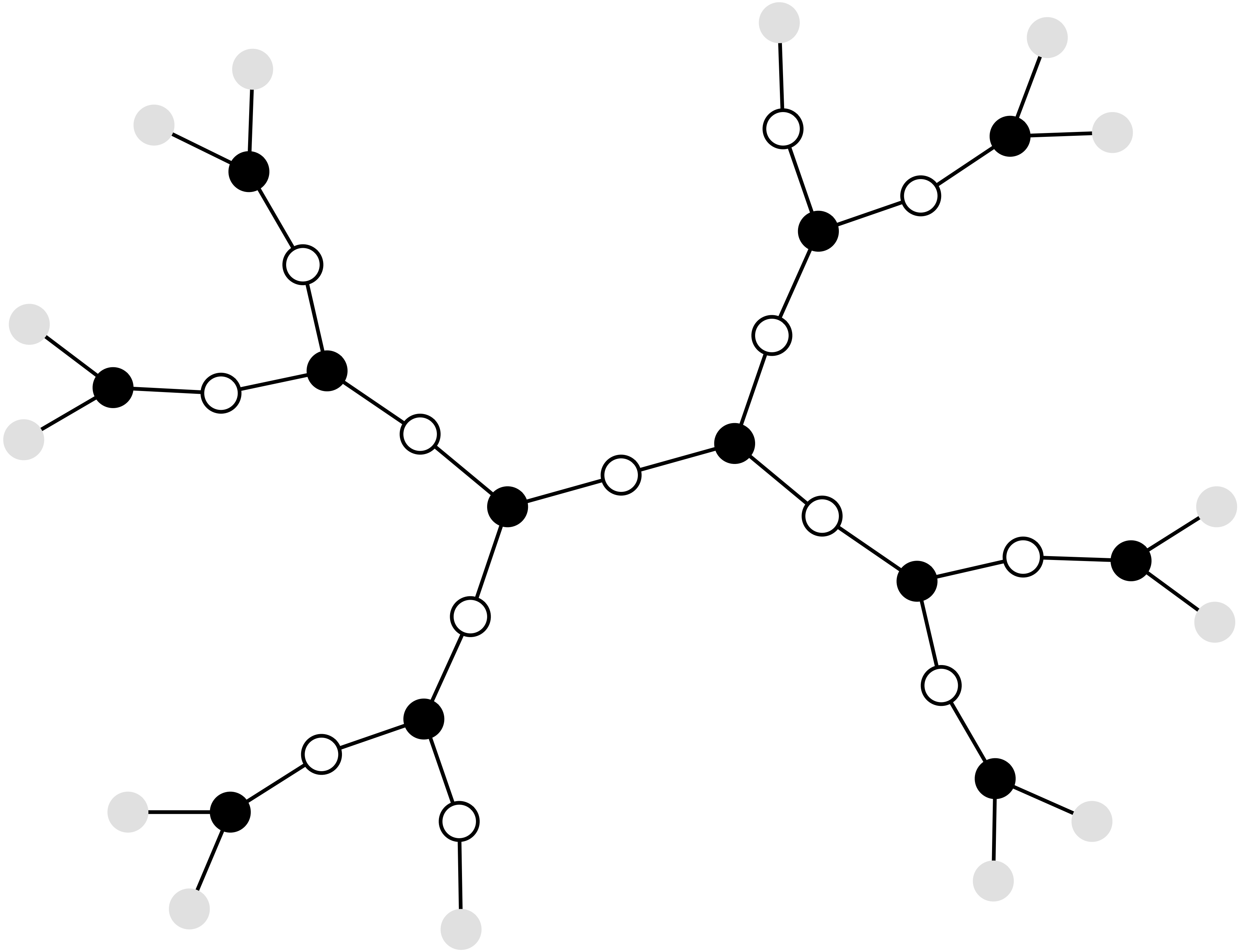
Round elimination turns problem X_0 into a new problem X_1 that **can be solved 1 round faster**

Round elimination turns problem X_0 into a new problem X_1 that **can be solved 1 round faster**

Bipartite locally verifiable problems

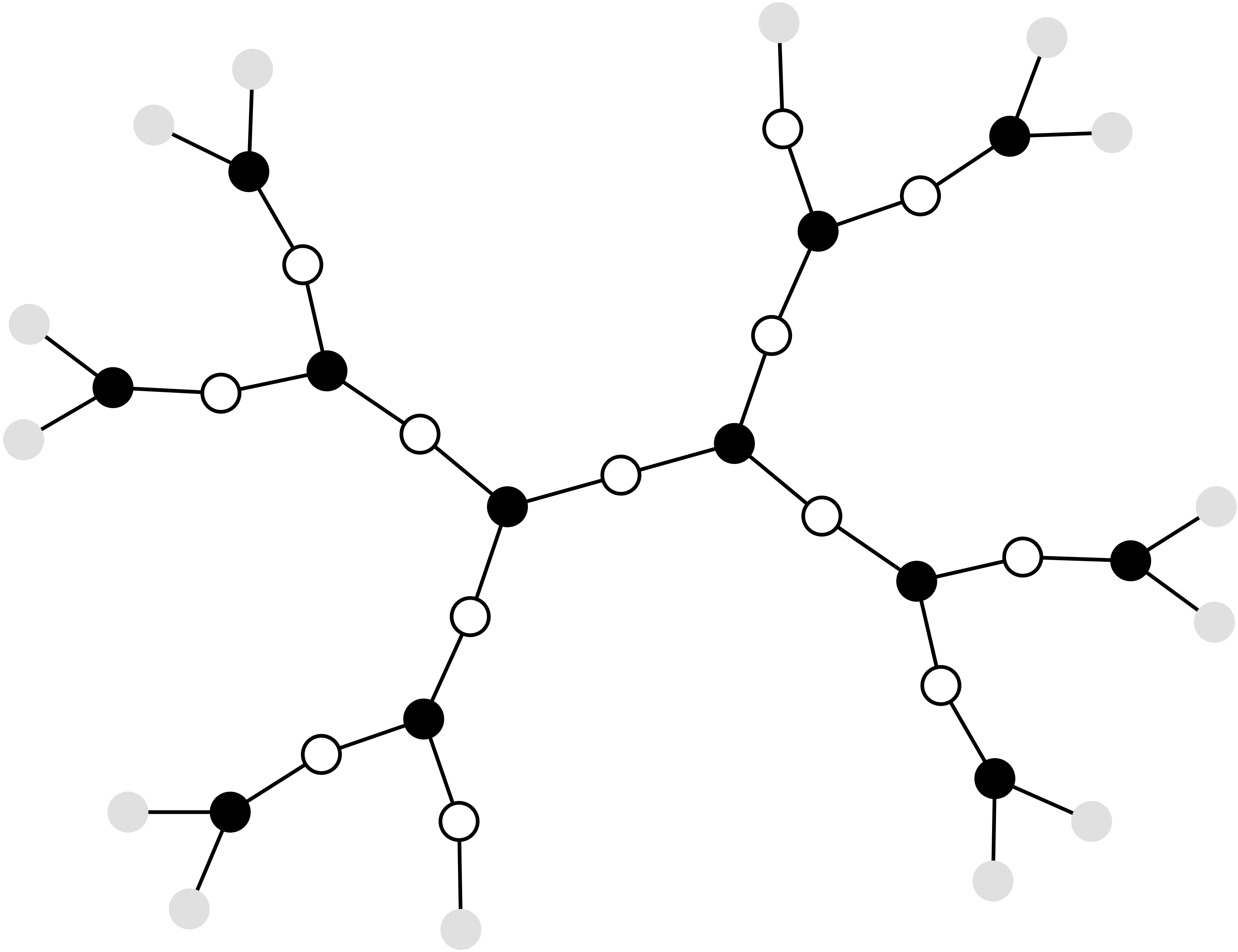
Toy example

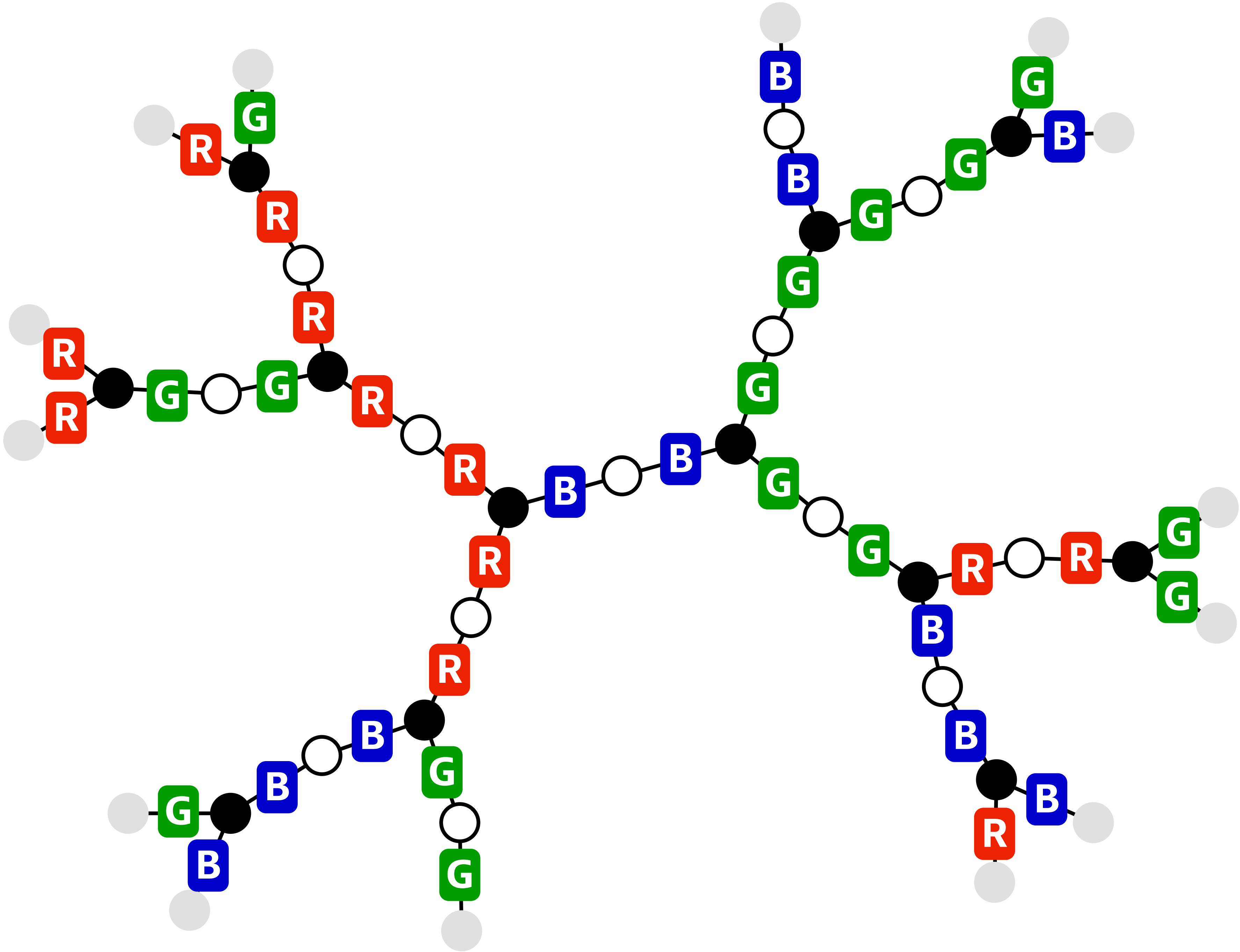
- **“Weak 3-labeling”** in 3-regular graphs
- **Goal:**
  - label edges with **R**, **G**, **B**
  - each node incident to at least two different colors
- **First step:** encode this as a bipartite locally verifiable problem



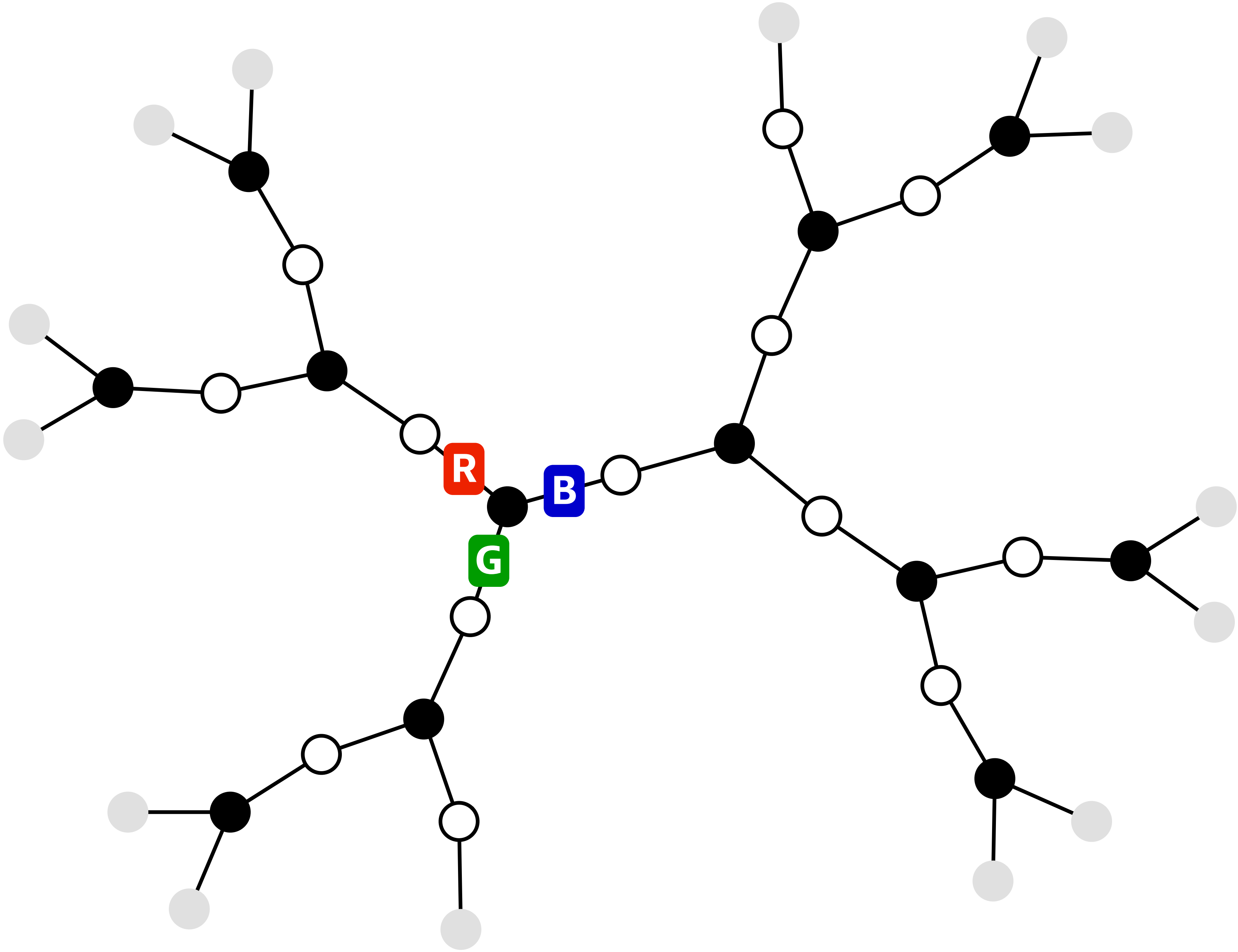
Weak
3-labeling

Weak 3-labeling





Weak 3-labeling



Weak
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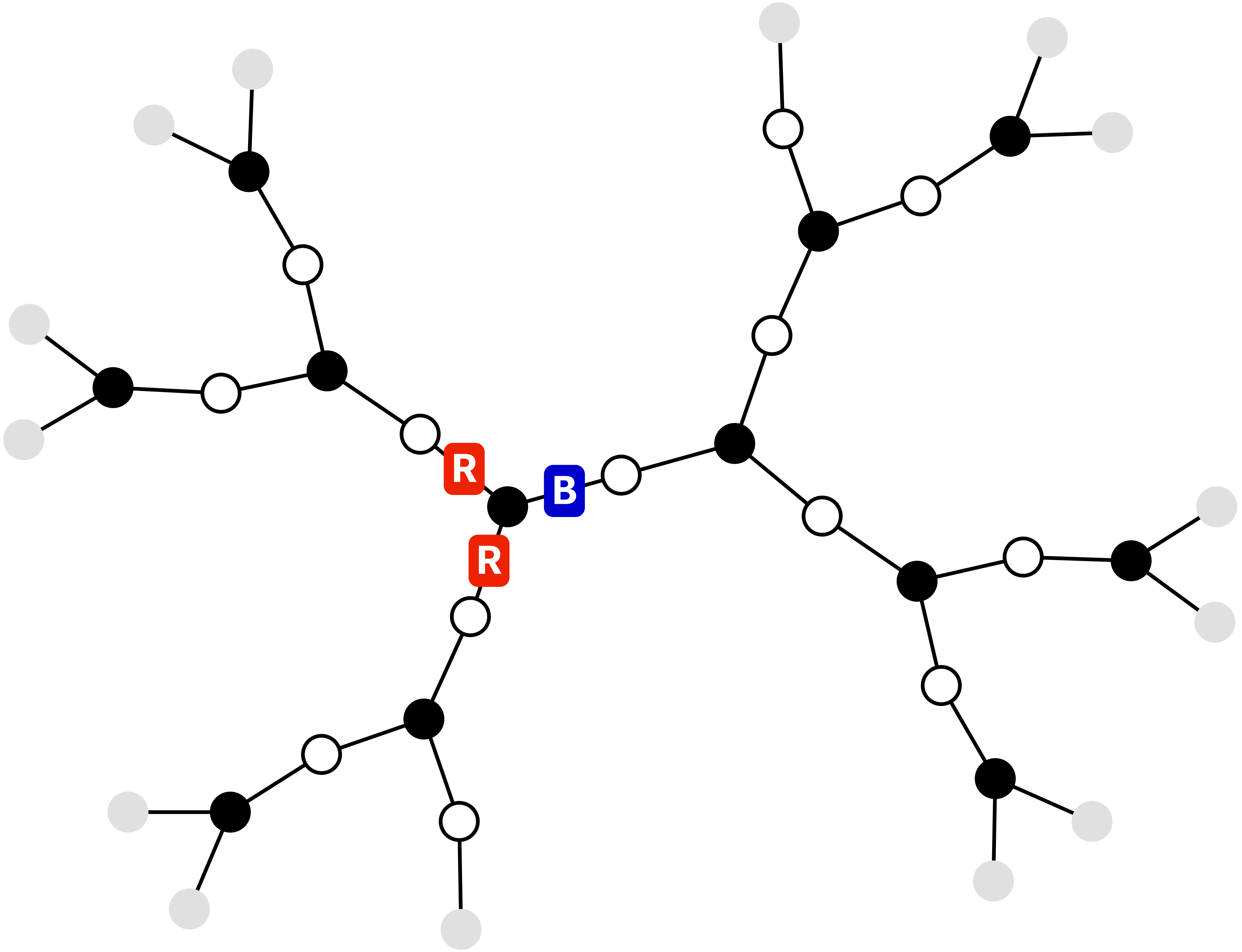
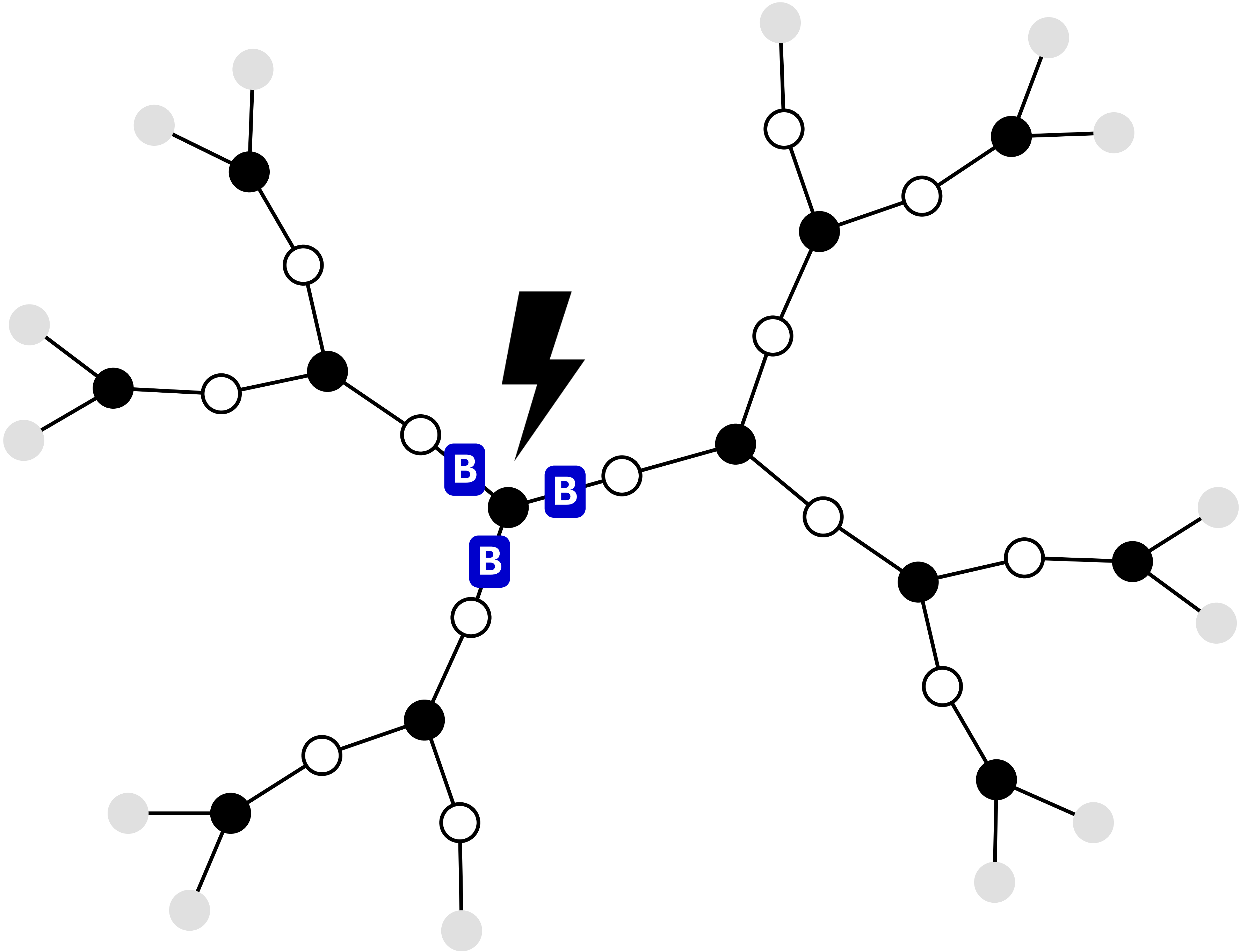
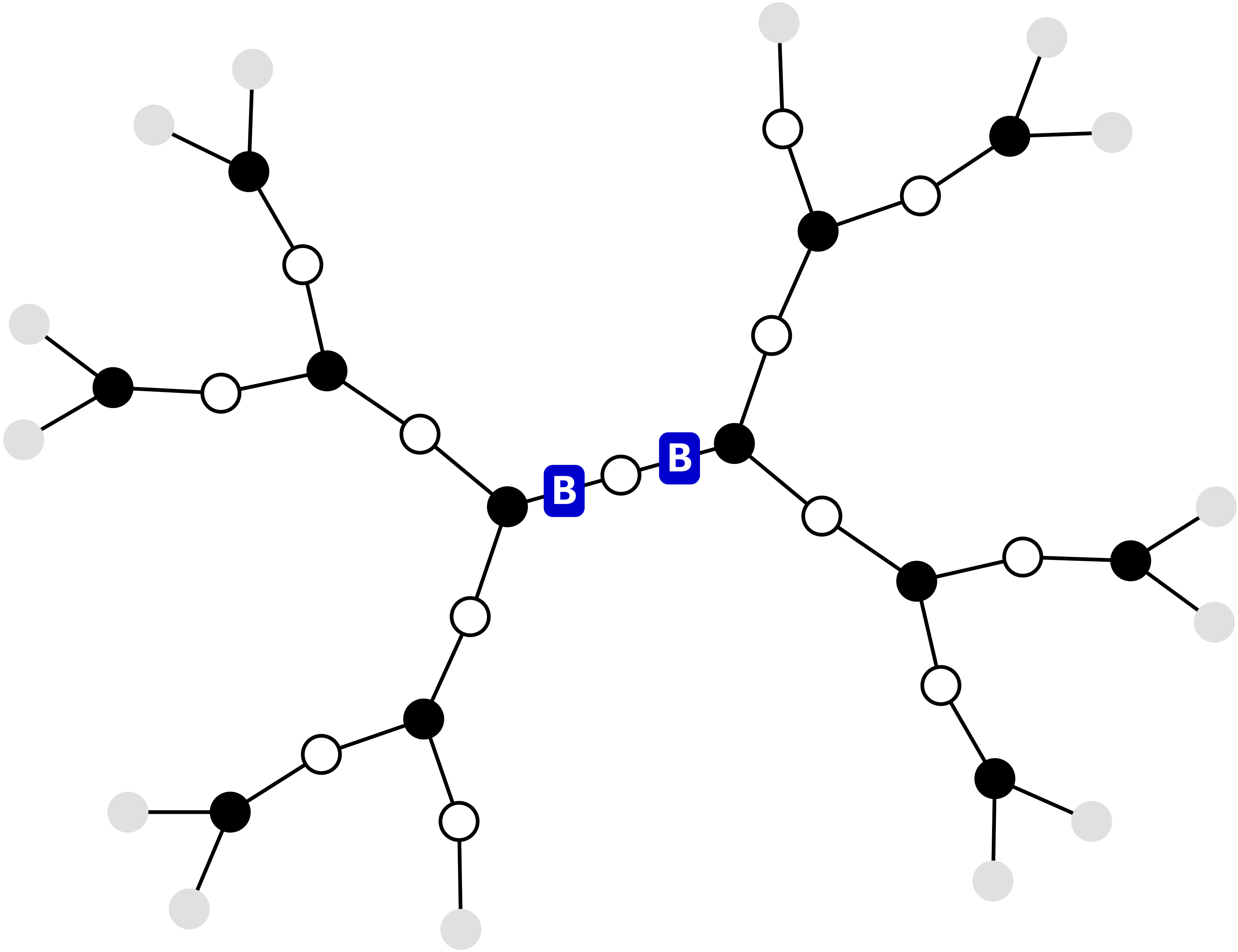
Weak 3-labeling

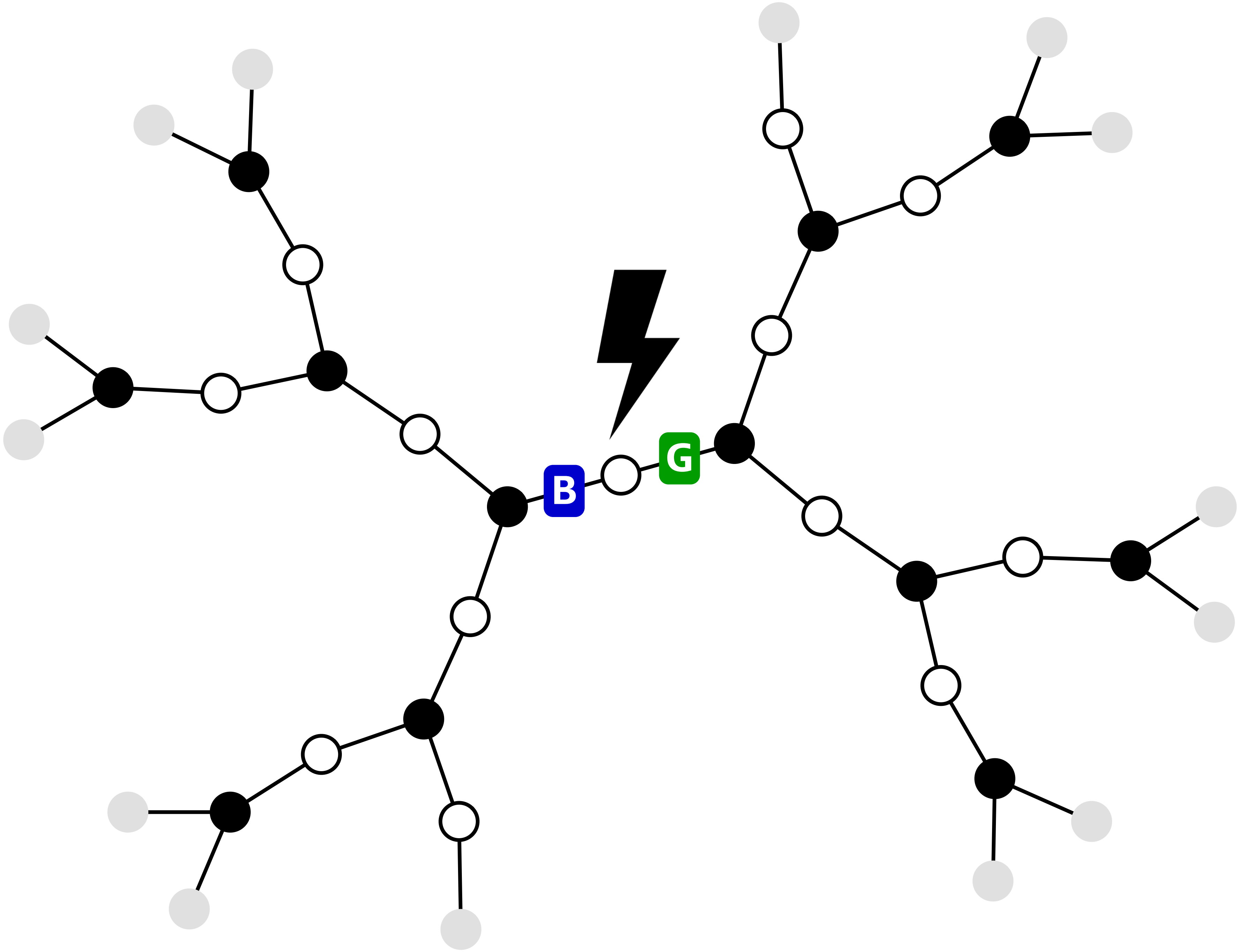




Weak 3-labeling

Weak 3-labeling





Weak 3-labeling

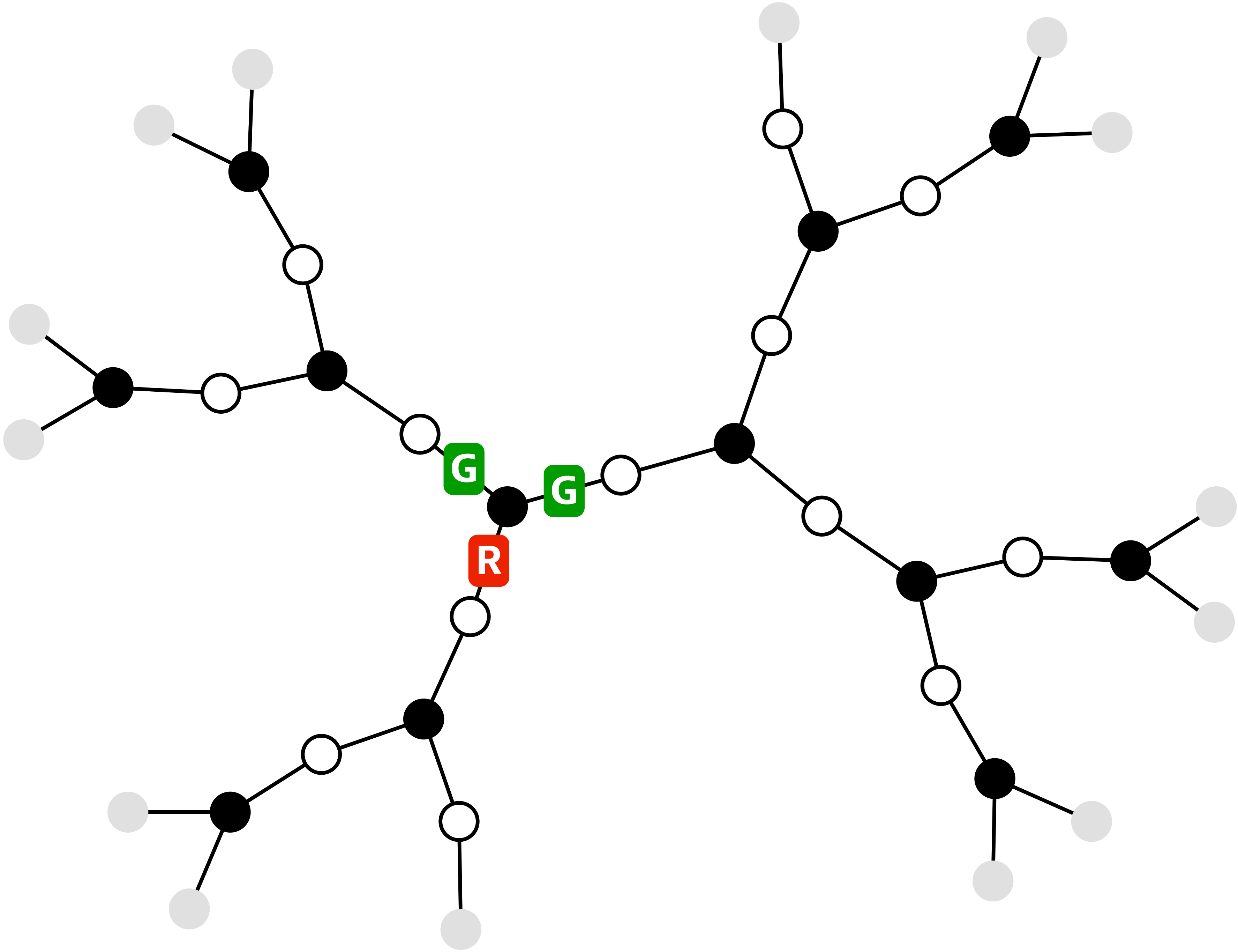
$\boldsymbol{X_0}$**:** labels **R**, **G**, **B**

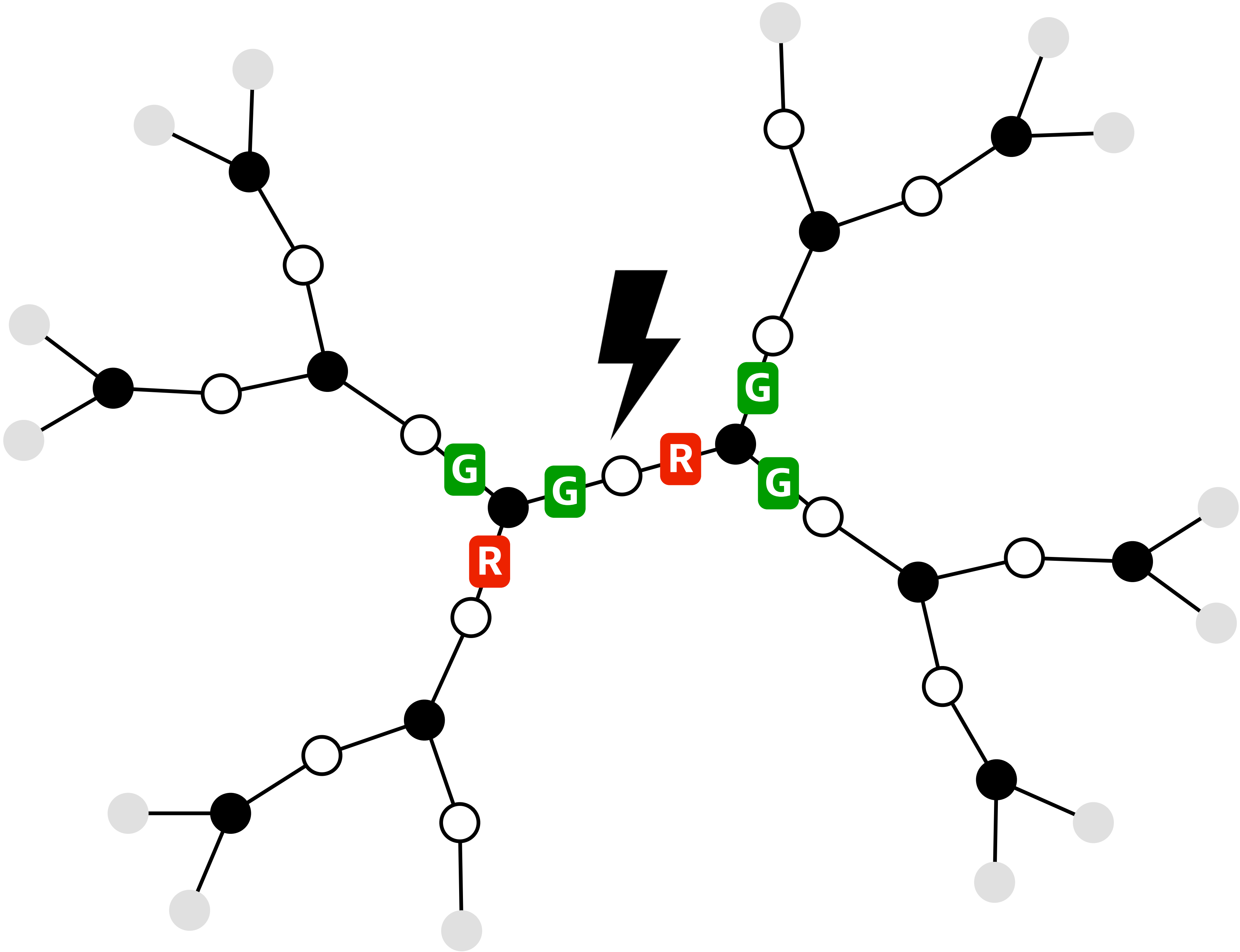
- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

X_0: labels **R**, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

Not solvable
in 0 rounds


G
G
R


G
G
G
G
R
R

X_0: labels R, G, B

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

$X_1 = \mathrm{re}(X_0)$:

X_0: labels **R**, **G**, **B**

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- passive (deg 2): equality

X_1 = re(X_0): labels **R**, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

X_0: labels **R**, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
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$X_1 = \mathrm{re}(X_0)$: labels **R**, **G**, **B**

- active (deg 2): equality
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Not solvable
in 0 rounds

X_0: labels **R**, **G**, **B**

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- passive (deg 2): equality

$X_1 = \mathrm{re}(X_0)$: labels **R**, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

Not solvable in 1 round

Not solvable in 0 rounds

X_0: **labels **R, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

$X_1 = \mathrm{re}(X_0)$: **labels **R, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

$X_2 = \mathrm{re}(X_1)$:

X_0: ** labels **R, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

X_1 = re(X_0): ** labels **R, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

X_2 = re(X_1): ** labels **R, **G**, **B**, **RG**, **RB**, **GB**, **RGB**

X_0: ** labels **R, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

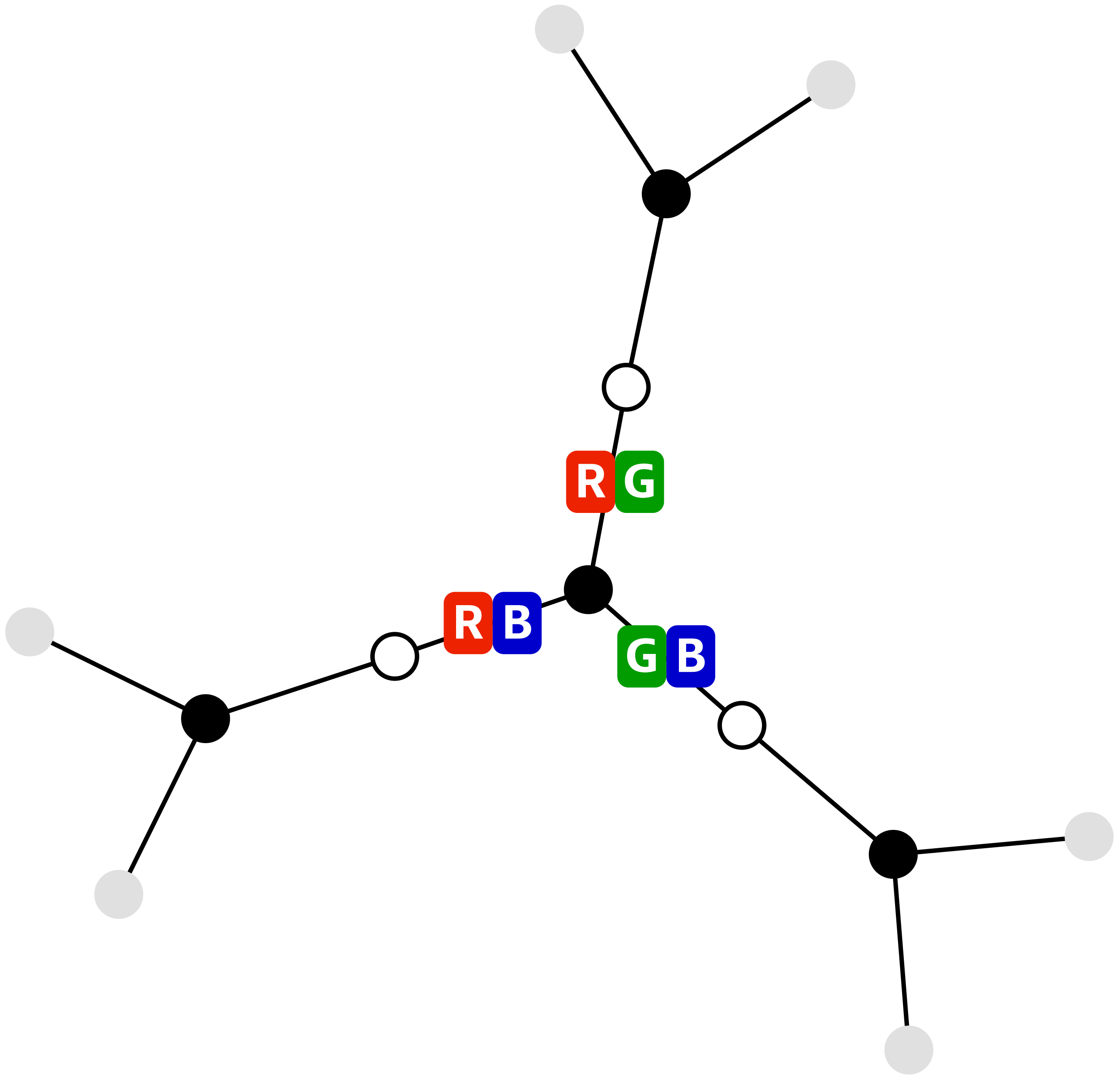
X_1 = re(X_0): ** labels **R, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

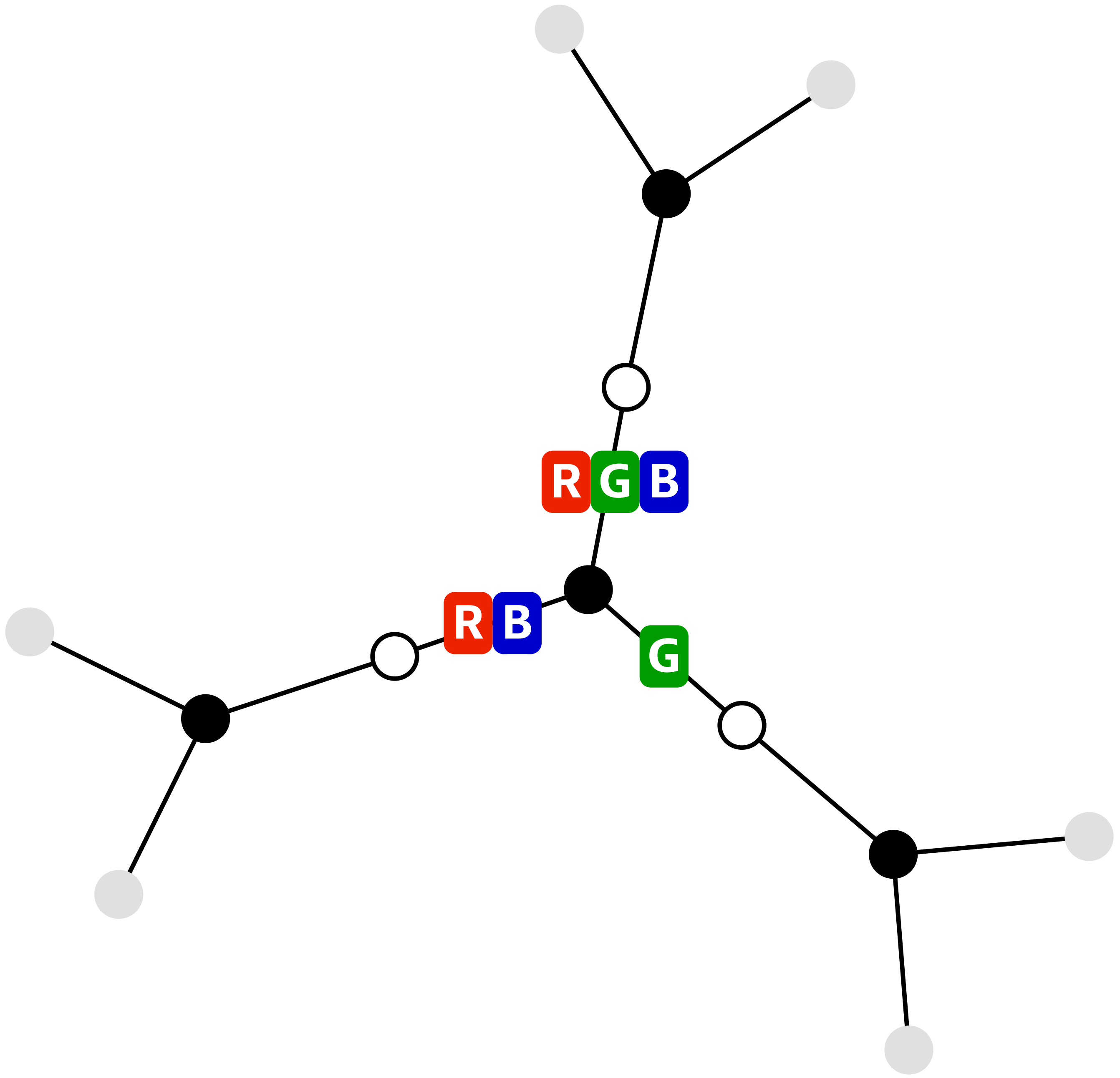
X_2 = re(X_1): ** labels **R, **G**, **B**, **RG**, **RB**, **GB**, **RGB**

- active (deg 3): empty intersection

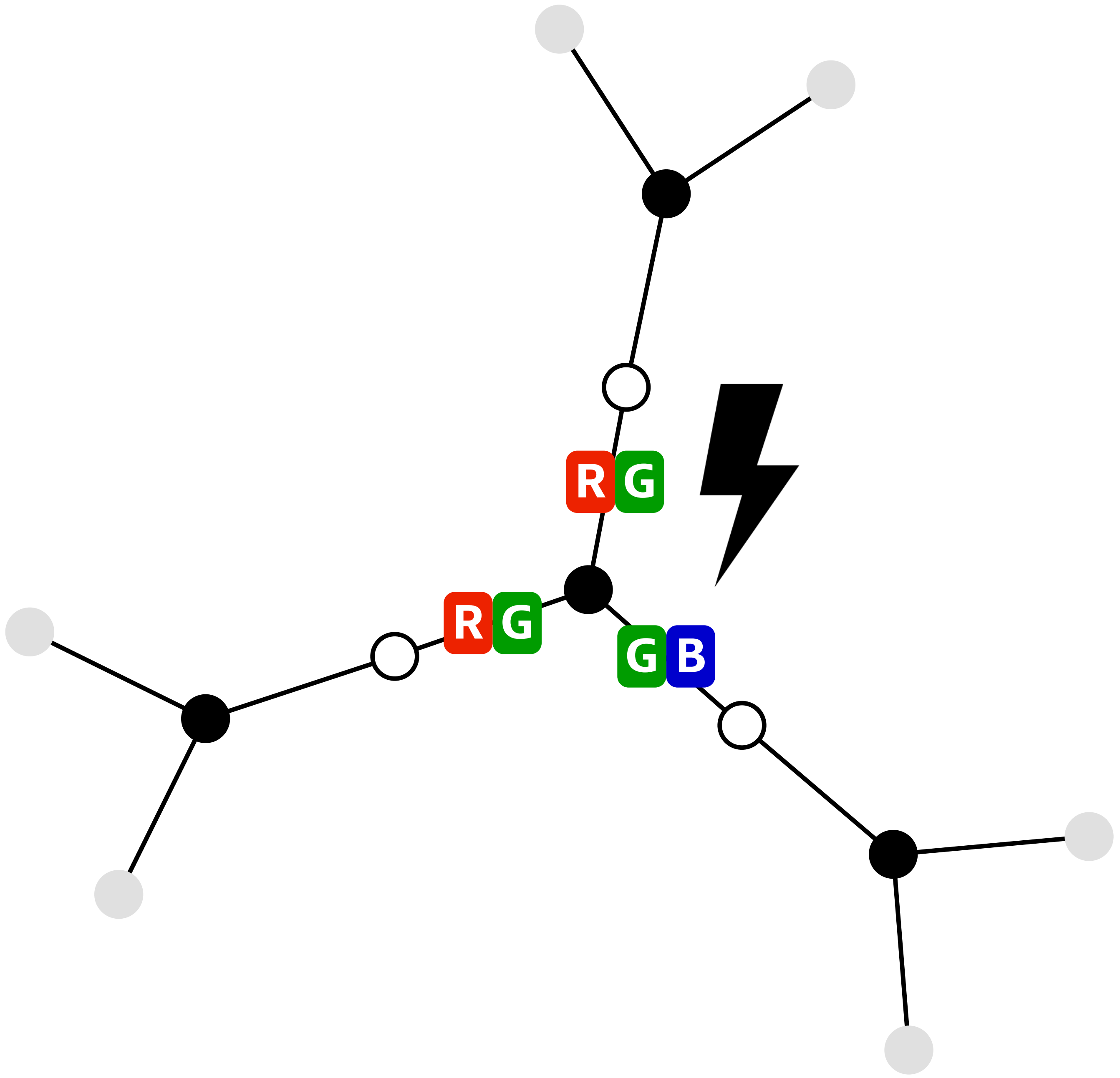
$$X_2 = \mathrm{re}(\mathrm{re}(X_0))$$



$X_2 = \mathrm{re}(\mathrm{re}(X_0))$



$$X_2 = \mathrm{re}(\mathrm{re}(X_0))$$



X_0: labels R, G, B

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

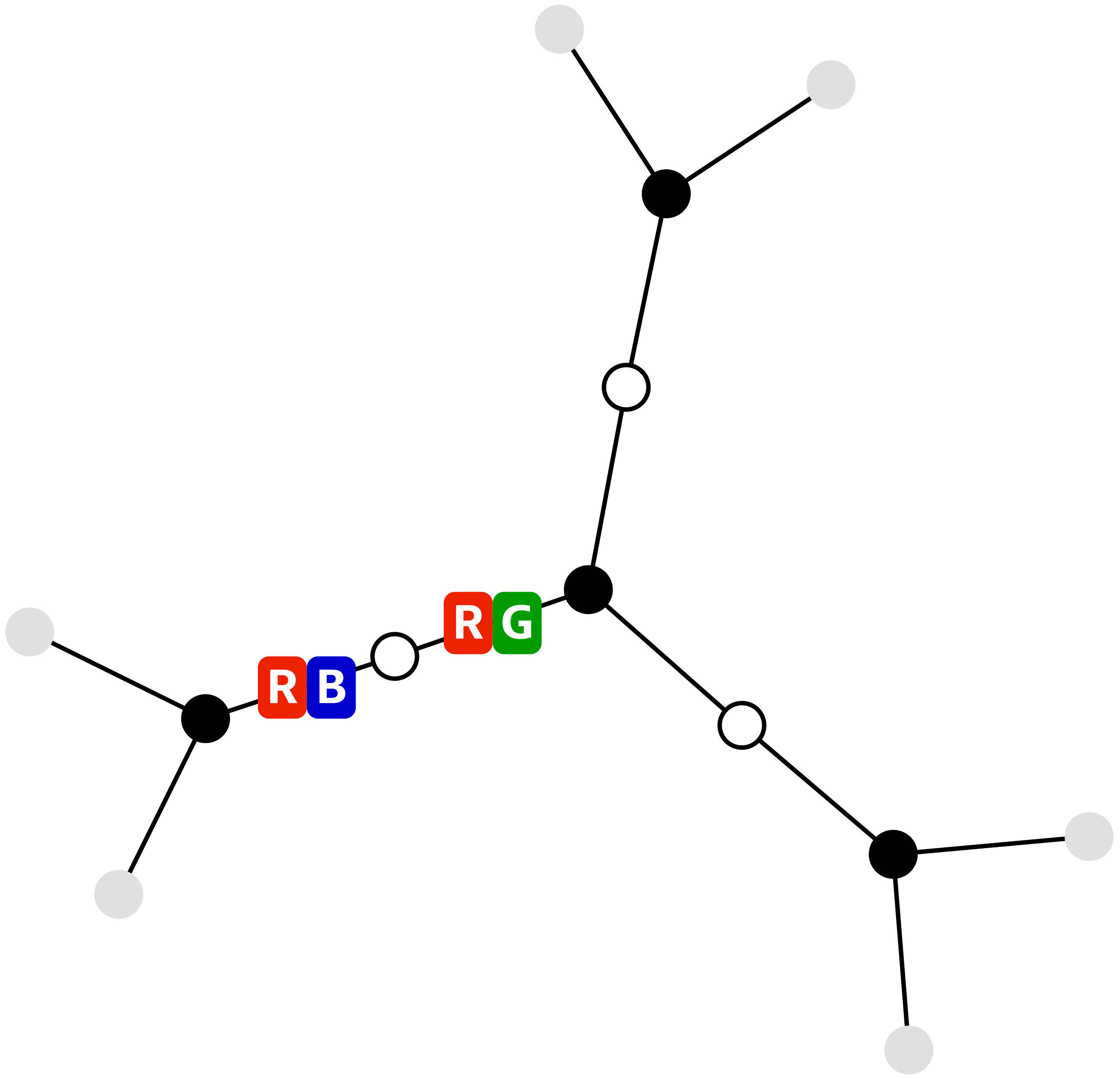
$X_1 = \mathrm{re}(X_0)$: labels R, G, B

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

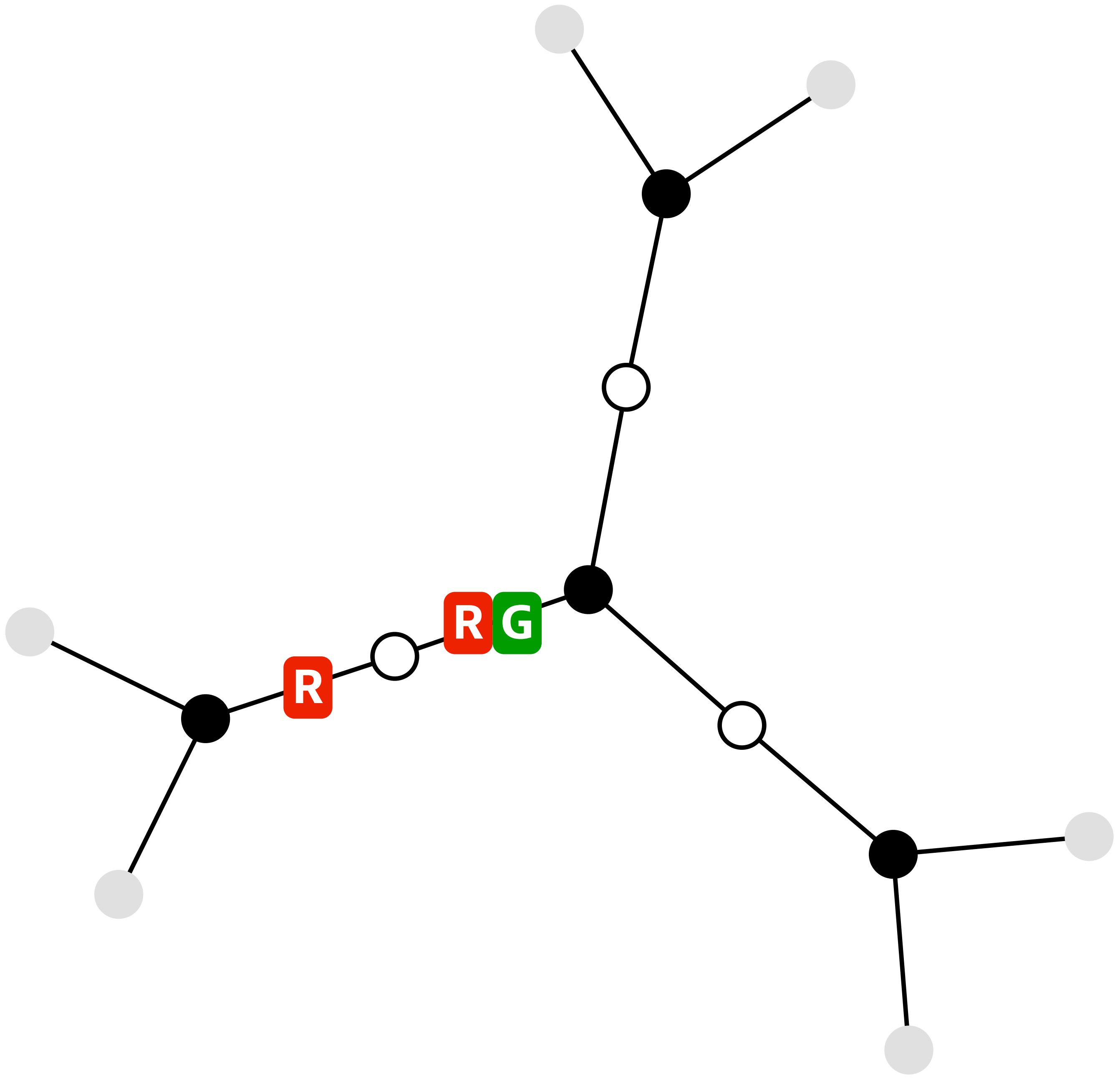
$X_2 = \mathrm{re}(X_1)$: labels R, G, B, RG, RB, GB, RGB

- active (deg 3): empty intersection
- passive (deg 2): non-empty intersection

$$X_2 = \mathrm{re}(\mathrm{re}(X_0))$$



$$X_2 = \mathrm{re}(\mathrm{re}(X_0))$$



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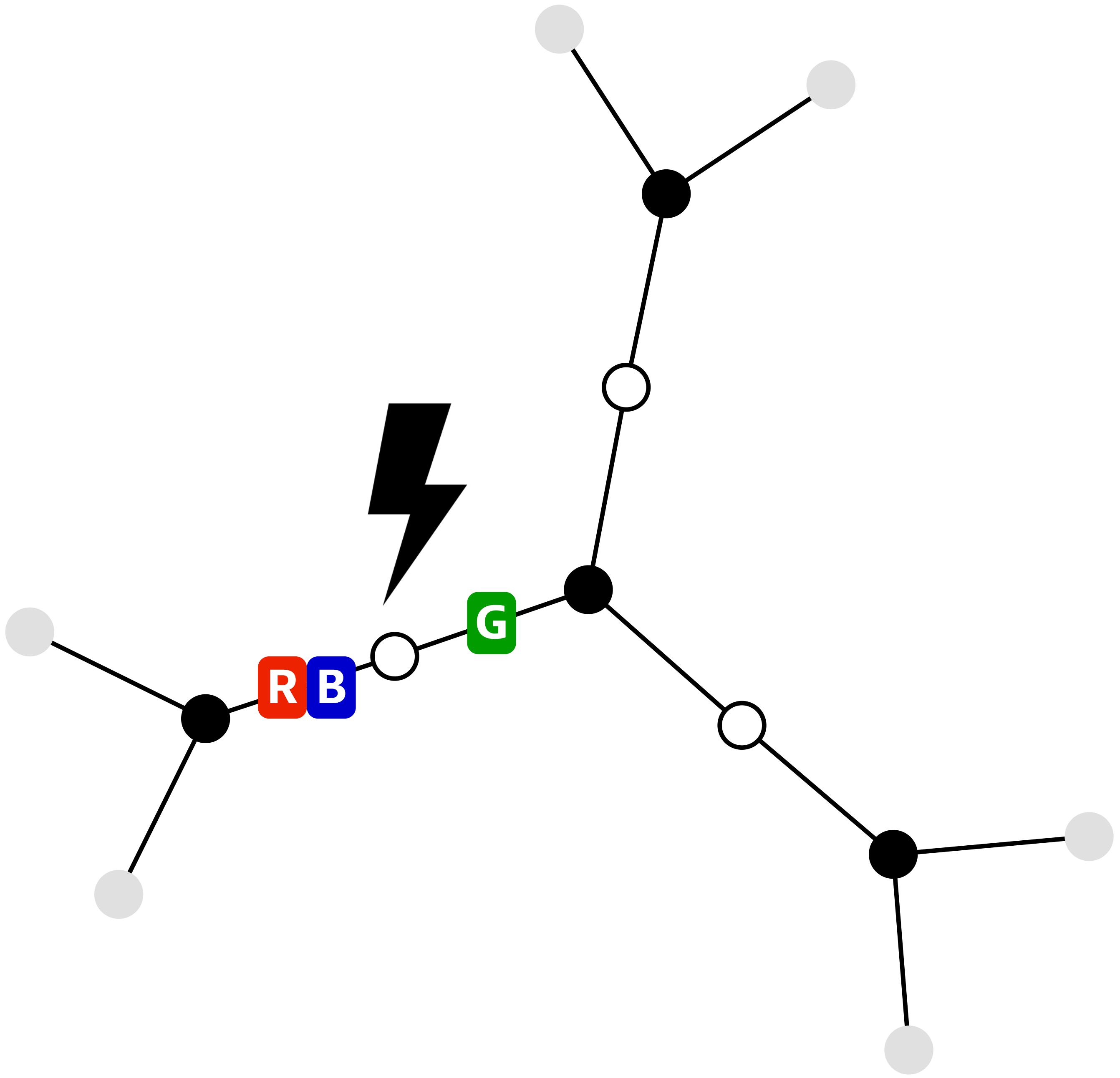

X_0: labels **R**, **G**, **B**

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X_1 = re(X_0): labels **R**, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

X_2 = re(X_1): labels **R**, **G**, **B**, **RG**, **RB**, **GB**, **RGB**

- active (deg 3): empty intersection
- passive (deg 2): non-empty intersection

Solvable
in 0 rounds

X_0: labels R, G, B

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

$T = 2$

X_1 = re(X_0): labels **R**, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

$T = 1$

X_2 = re(X_1): labels **R**, **G**, **B**, **RG**, **RB**, **GB**, **RGB**

- active (deg 3): empty intersection
- passive (deg 2): non-empty intersection

$T = 0$

Summary

- Meta-algorithm: **round elimination**
- Many recent lower bounds are based on RE
- Also used to show **quantum advantage**
  - ***by construction:*** easy for quantum
  - ***by RE:*** hard for classical
- Open: when is it "complete"?