The Light Bulb Problem

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The Light Bulb Problem (Valiant 1988)

Given as input n Boolean vectors of length d with the promise that the vectors are uniform random except for two vectors that are $> \rho$-correlated, how quickly can one find the correlated pair?

(The correlation of two vectors $x, y \in \{-1, 1\}^d$ is $\frac{1}{d}\langle x, y\rangle = \frac{1}{d}\sum_{i=1}^{d} x_i y_i$.)

Example with Matlab

```
n = 200;                                  % number of light bulbs to observe
rho = 0.7;                                % planted correlation strength
d = ceil(5/(rho*rho)*log(n));             % dimension to get unique pair whp
ij = randperm(n,2);                       % the planted pair (i,j)
a = 2*round(rand(n,d))-1;                 % generate the random background
m = ones(1,d);                            % plant the correlation ...
m(randperm(d,floor(d*(1-rho)/2)))=-1;     % ... flip coordinates with -1 in m
a(ij(1),:)=a(ij(2),:).*m;                 % ... insert correlation between (i,j)

imshow(a)                                 % display the data
```

Uniqueness of the correlated pair via Hoeffding's inequality

- **Hoeffding (1963).** Let $Z_1, Z_2, \ldots, Z_d$ be independent random variables with $a_i \leq Z_i \leq b_i$ for $i = 1, 2, \ldots, d$ and let $S_d = Z_1 + Z_2 + \ldots + Z_d$. Then,

$$\Pr\left(|S_d - \mathbb{E}[S_d]| \geq t\right) \leq 2\exp\left(\frac{-2t^2}{\sum_{i=1}^{d}(b_i - a_i)^2}\right)$$

- For $Z_1, Z_2, \ldots, Z_d \in \{-1, 1\}$ uniform random and $\rho \geq 0$, we thus have

$$\Pr\left(|S_d| \geq \rho d\right) \leq 2\exp\left(-\frac{\rho^2 d}{2}\right)$$

- For $d \geq 5\rho^{-2}\log n$ we have $\Pr\left(|S_d| \geq \rho d\right) \leq 2n^{-5/2}$, which by the union bound on the $\binom{n}{2}$ pairs of vectors implies the ρ-correlated pair is unique with probability at least $1 - O(n^{-1/2})$

An incomplete bibliography on the light bulb problem

- Valiant [7]
- Paturi, Rajasekaran, and Reif [5]
- Valiant [6]
- Karppa, Kaski, and Kohonen [3]
- Karppa, Kaski, Kohonen, and Ó Catháin [4]
- Alman [1]
- Alman and Zhang [2]

References I

[1] J. Alman, An illuminating algorithm for the light bulb problem, in *2nd Symposium on Simplicity in Algorithms, SOSA 2019, January 8-9, 2019, San Diego, CA, USA* (J. T. Fineman and M. Mitzenmacher, Eds.). Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019, pp. 2:1–2:11. [doi:10.4230/OASICS.SOSA.2019.2].

[2] J. Alman and H. Zhang, Generalizations of matrix multiplication can solve the light bulb problem, in *64th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2023, Santa Cruz, CA, USA, November 6-9, 2023*. IEEE, 2023, pp. 1471–1495. [doi:10.1109/FOCS57990.2023.00090].

[3] M. Karppa, P. Kaski, and J. Kohonen, A faster subquadratic algorithm for finding outlier correlations, *ACM Trans. Algorithms* 14 (2018), 31:1–31:26. [doi:10.1145/3174804].

References II

[4] M. Karppa, P. Kaski, J. Kohonen, and P. Ó Catháin, Explicit correlation amplifiers for finding outlier correlations in deterministic subquadratic time, *Algorithmica* 82 (2020), 3306–3337.
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[doi:10.1006/INCO.1995.1038].

[6] G. Valiant, Finding correlations in subquadratic time, with applications to learning parities and the closest pair problem, *J. ACM* 62 (2015), 13:1–13:45.
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[7] L. G. Valiant, Functionality in neural nets, in *Proceedings of the First Annual Workshop on Computational Learning Theory, COLT '88, Cambridge, MA, USA, August 3-5, 1988* (D. Haussler and L. Pitt, Eds.). ACM/MIT, 1988, pp. 28–39.