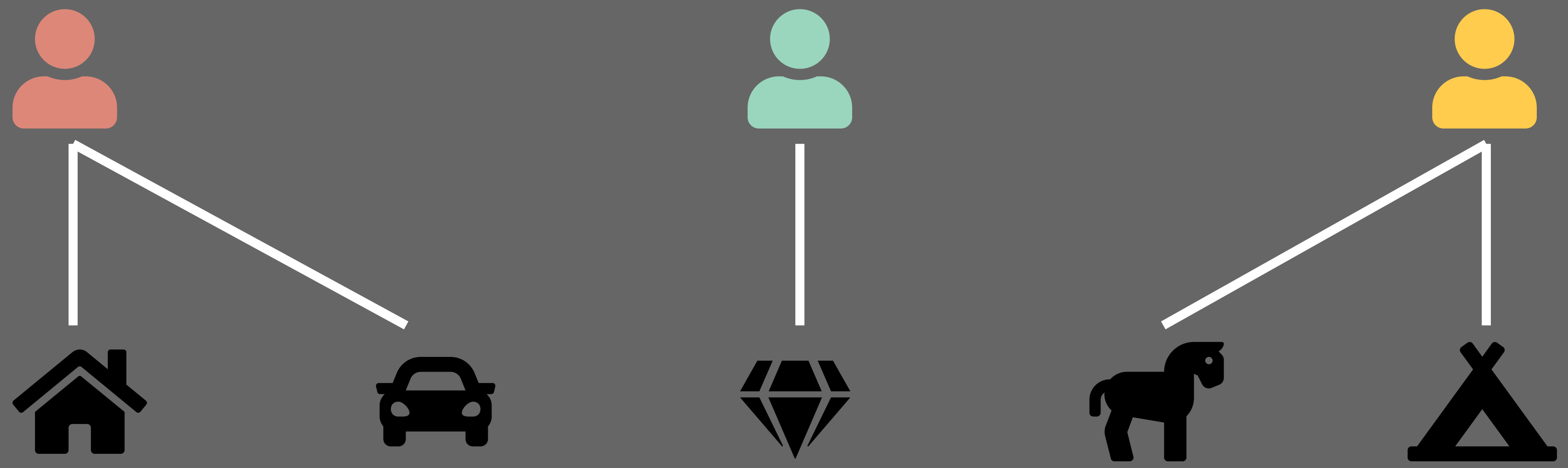
	10	5	20	14	9
	7	7	10	9	9
	11	6	30	10	5



Fair Division of Indivisible Goods

Exploring “envy-freeness up to any good” (EFX)

Frontier Friday · March 14, 2025 · Corinna Coupette

Problem Setting

Set M of m indivisible goods

Set V of n agents with valuation functions $v_i : 2^M \to \mathbb{R}_{\geq 0}$

Restrictions: binary, *additive*, submodular, ...

Goal: *Fair* partition of M into n bundles $\langle X_1, \ldots, X_n \rangle$

Fairness notion: EFX

$v_i(X_i) \geq v_i(X_j \setminus \{g\})$, all $j \neq i$, all $g \in X_j$ (remove *least* valuable good)

Do EFX allocations always exist?

Yes for 2 agents, 3 additive agents, α-EFX with few* unallocated goods, ...

Connection to Extremal Combinatorics

Given an integer $d > 0$, the *rainbow cycle number* $R(d)$ is the largest k s.t.

there exists a k-partite graph $G = (V_1 \cup \cdots \cup V_k, E)$ with

$1 \leq |V_i| \leq d$ for each $i \in [k]$

each $v \in V_i$ has exactly one incoming edge from each V_j for all $j \neq i$

there is no rainbow cycle (at most one vertex from each part)

Let $h^{-1}(d)$ be the smallest integer ℓ s.t. $h(\ell) = \ell \cdot R(\ell) \geq d$. Then:

There exists a $(1-\varepsilon)$-EFX allocation with $O\left(\frac{n}{\varepsilon h^{-1}(n/\varepsilon)}\right)$ unallocated goods.

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