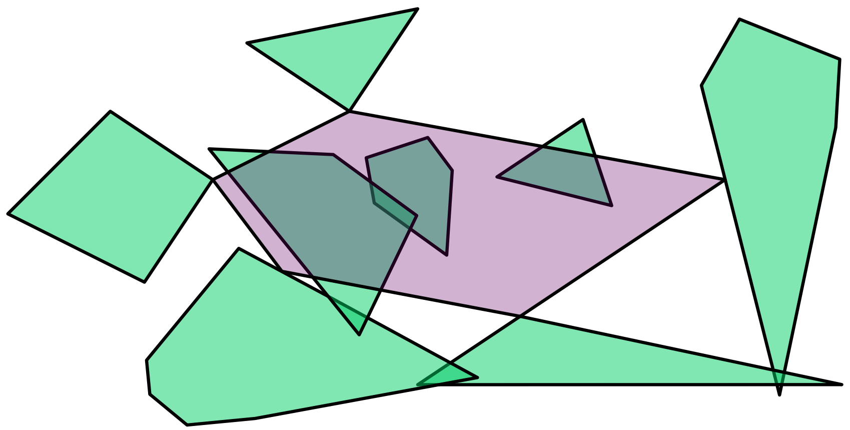
Minimum convex intersecting polytope



HALT Frontier Friday, 9 May 2025

Aalto University
School of Science

“Smallest” intersecting convex polygon of objects

The problem

Given n objects in $\mathbb{R}^d$, find the “smallest” convex polytope that intersects all objects.

"Smallest" intersecting convex polygon of objects

$\mathbb{R}^2$: Minimum perimeter/ minimum area

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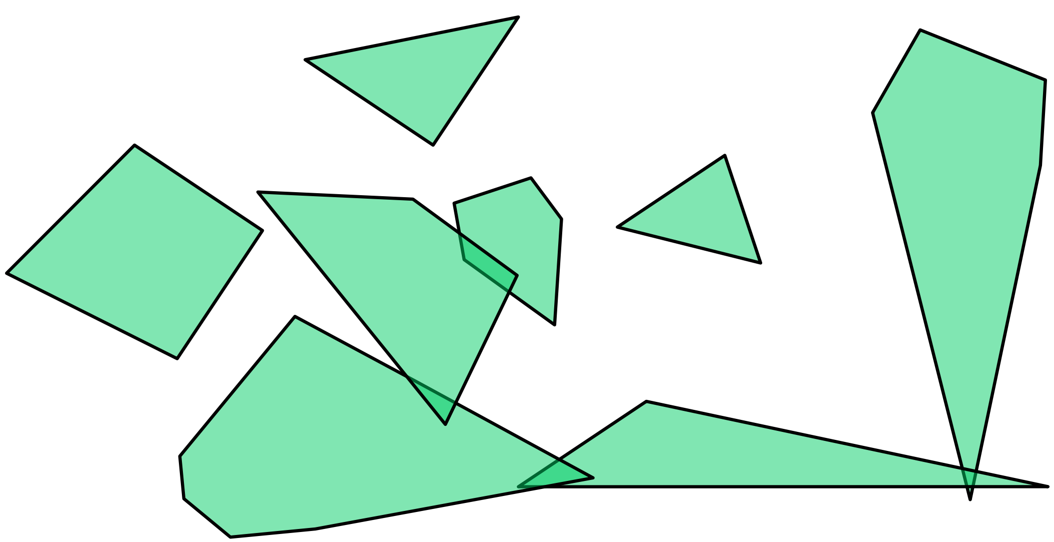

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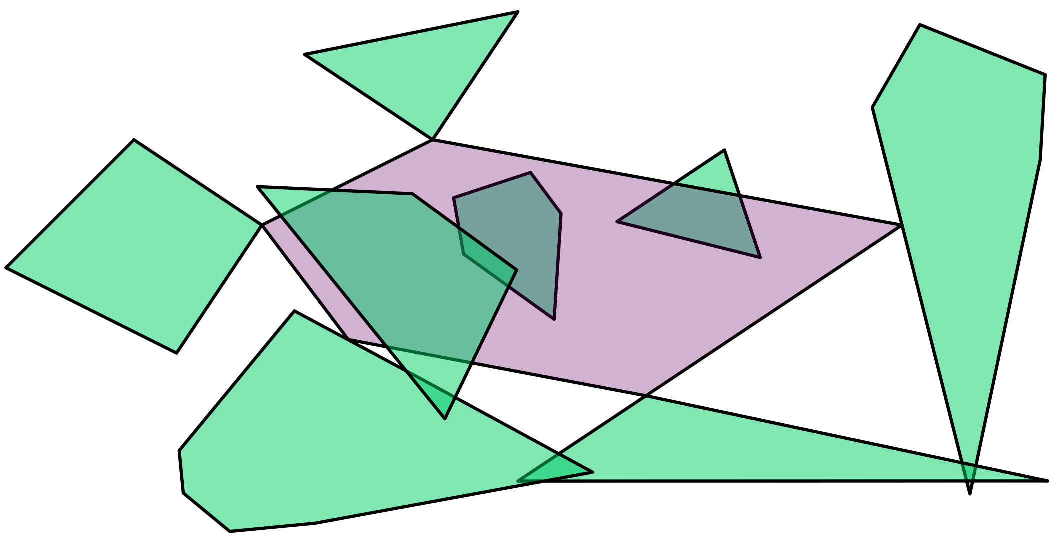

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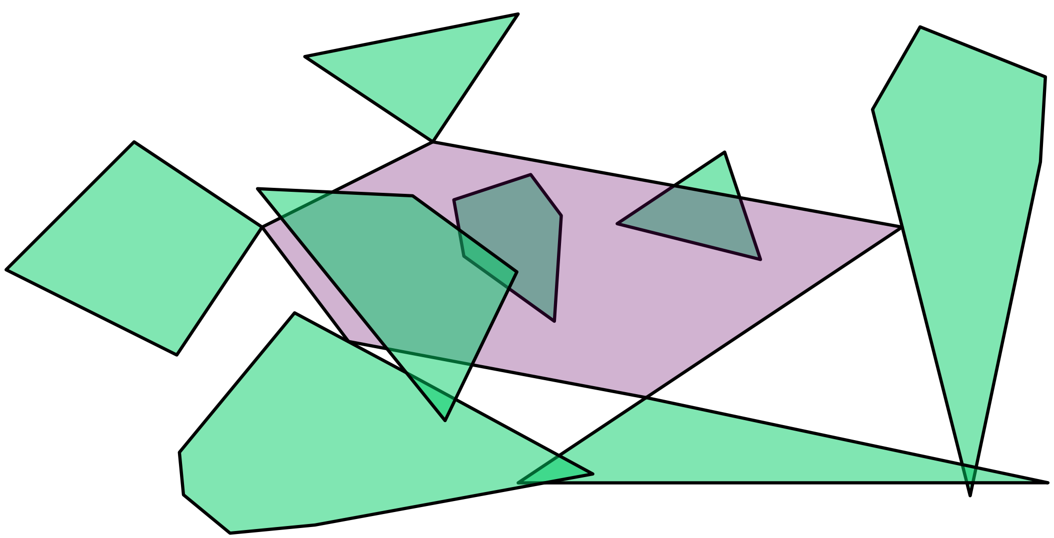

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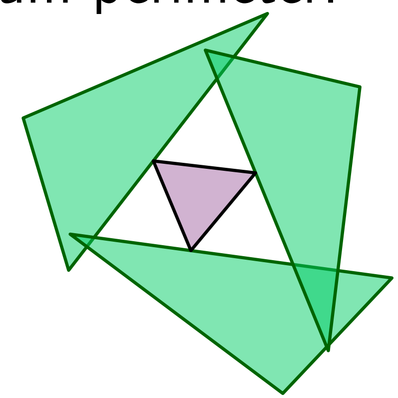

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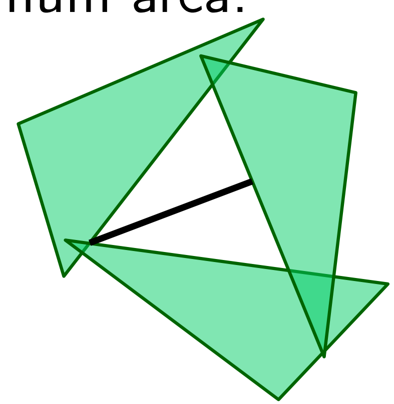

Related work: Generic PTAS and specialized exact

- van Kreveld and Löffler (2008, 2010):

  min/max perimeter/area convex hull of "imprecise points"

  parallel line segments, disjoint parallel squares ⟶ poly time

  max area/perimeter for segments ⟶ NP-hard

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- Dumitrescu and Jiang (2012)

min perimeter intersecting polygon for segments / convex polygons $\longrightarrow$ PTAS: $\frac{n^{O(1)}}{\varepsilon} + 2^{O(1/\varepsilon^{2/3})} n$

min perim. int. pol. of concave polygons $\longrightarrow$ NP-hard

Is the problem of computing a minimum-perimeter intersecting polygon of a set of segments NP-hard?

Other known results

Theorem

Given convex polygons of total complexity n and $\varepsilon > 0$, there is an $O(n^{\omega+o(1)}/\varepsilon + n/\varepsilon^8)$ time $(1+\varepsilon)$-approximation (i.e., FPTAS) for their minimum perimeter intersecting polygon. $(\omega < 2.373)$

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Given n line segments, there is an $O(n^9 \log n)$ time algorithm to compute their minimum perimeter intersecting polygon.

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Theorem

Given convex polygons of total complexity n and $\varepsilon > 0$, there is an $O(n^{17}\log(1/\varepsilon) + n^{11}/\varepsilon^{24})$ time $(1+\varepsilon)$-approximation (i.e., FPTAS) for their minimum area convex intersecting polygon.

Problem(s) to consider now

Problem 1. Given n pairs of points in $\mathbb{R}^2$, find the minimum perimeter or minimum area convex polygon that contains at least one point from each pair.

$\rightarrow$ NP-hard from vertex cover

Can we get O(1)-approximation? What about an (F)PTAS?

Problem 2. Given n pairs of points in $\mathbb{R}^2$ where each pair share either x- or y-coordinates, find the minimum perimeter/minimum area convex polygon that contains at least one point from each pair.

Is this poly-time solvable?

Problem 3. Given n axis-parallel unit squares in $\mathbb{R}^2$, find the minimum perimeter/minimum area convex polygon that covers at least half of each square.

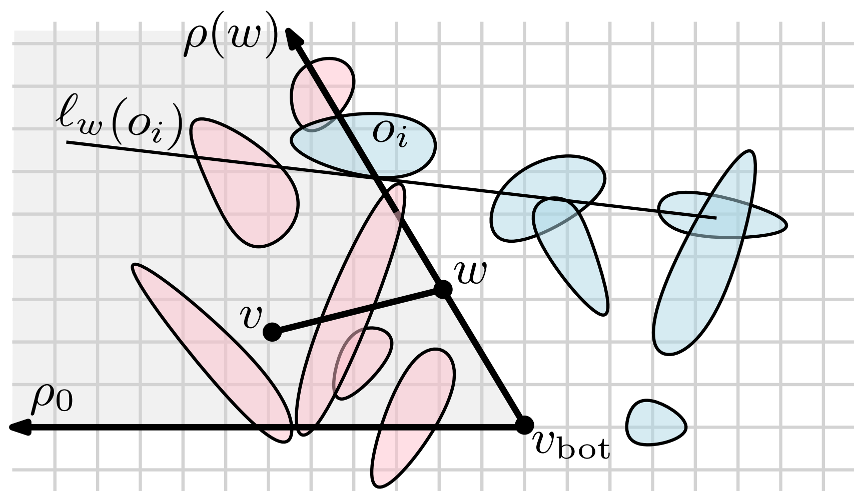
No idea about the complexity here! Stochastic approximation?

Problem 4. Given n convex objects in $\mathbb{R}^3$, find the minimum volume/minimum surface area convex polytope that intersects all objects.

Is this NP-hard?

What objects allow PTAS via coresets?

FPTAS for min perimeter intersecting polygon of convex objects



Step 1: Locating the optimum

Where is OPT
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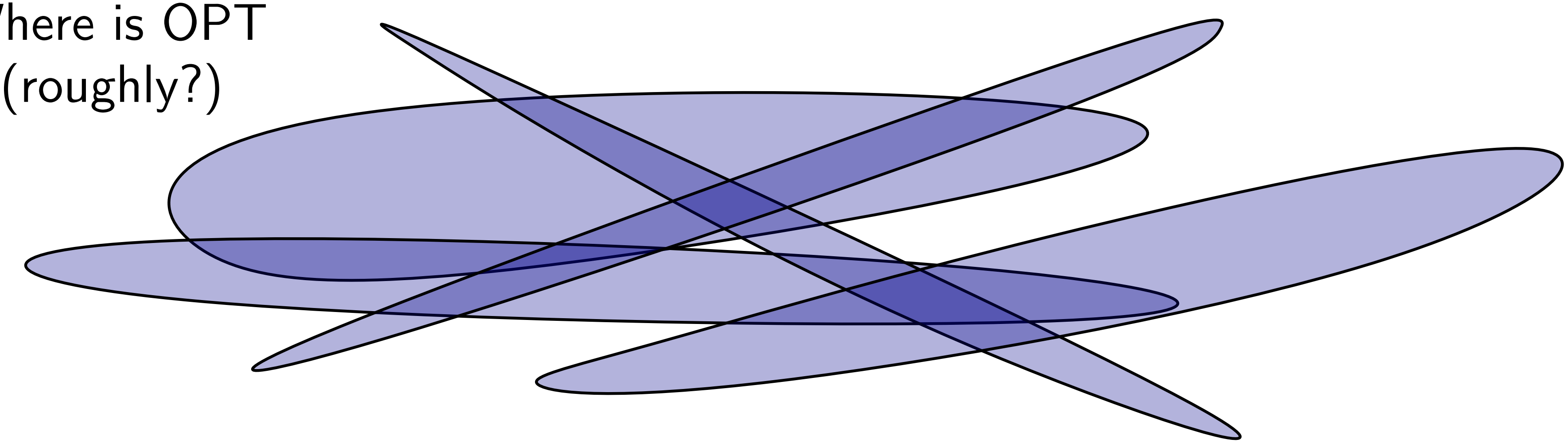

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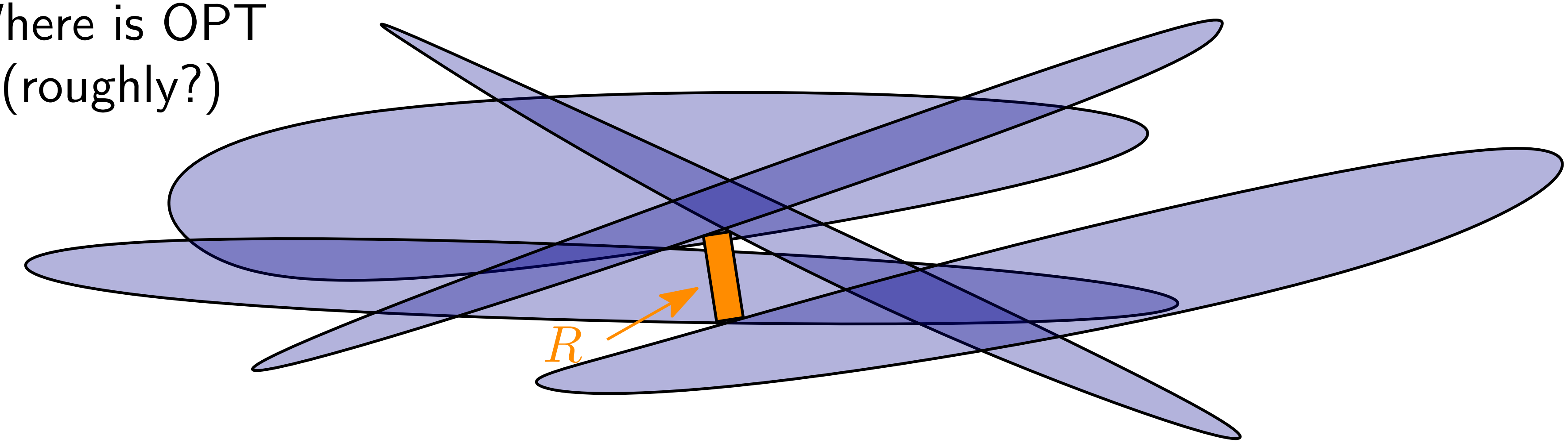

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The min bounding box of OPT is a constant-approximation, let's find it!

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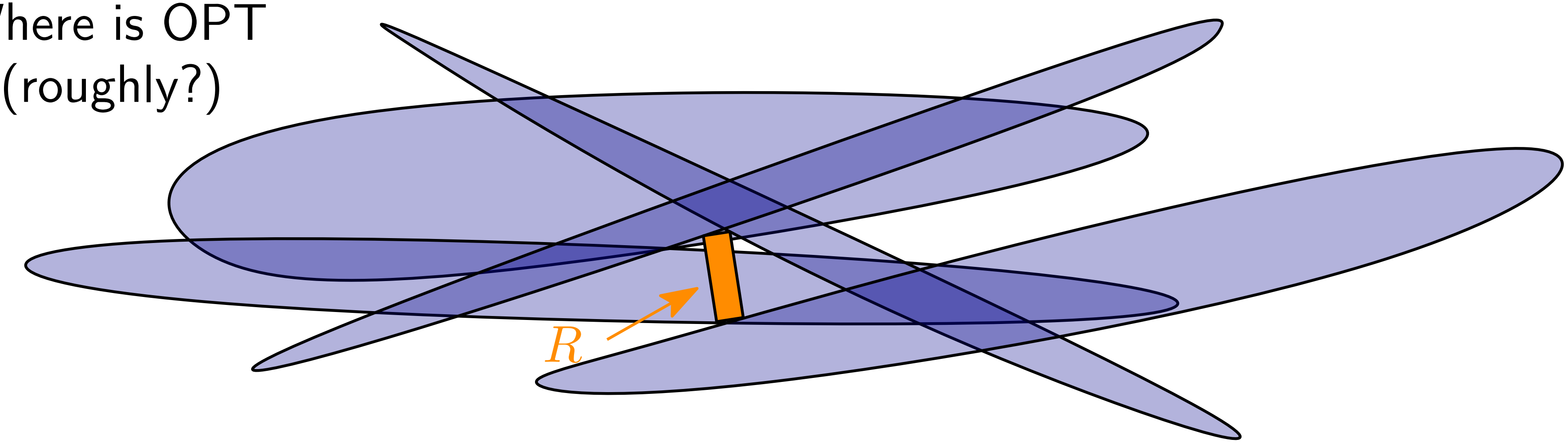

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For each orientation v among $O(1/\varepsilon)$ options:

- Compute the min perimeter feasible rectangle parallel to v. Solved with an LP.

Return the min perimeter rectangle R found so far

Will R cover some $(1+\varepsilon)$-approximation of OPT?

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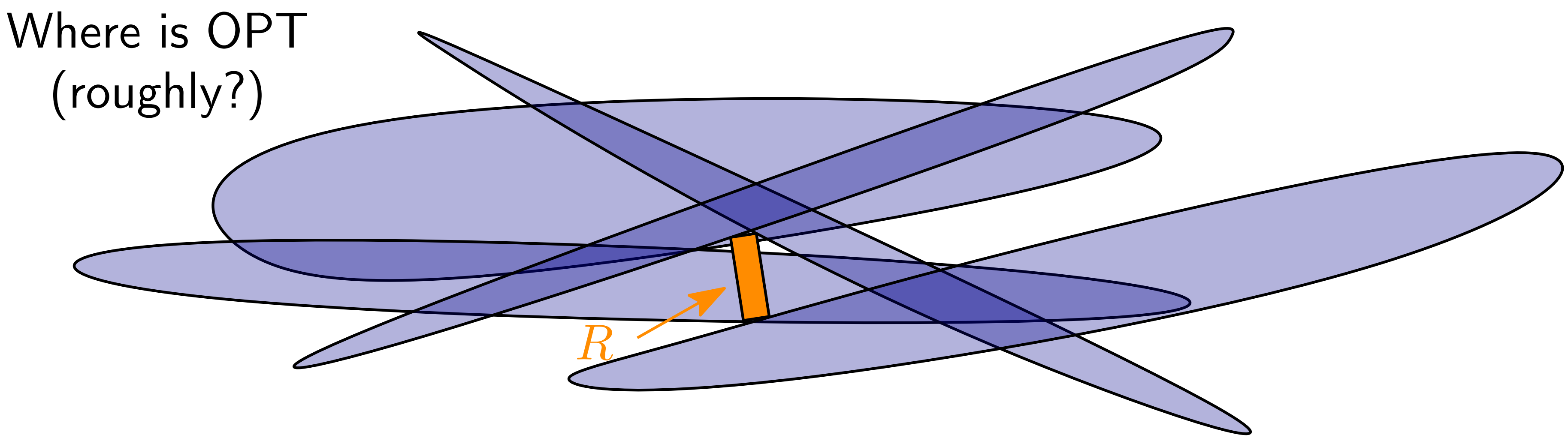

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Lemma (Dumitrescu and Jiang)
Either R is a $(1+\varepsilon)$-approximate solution, or there is an optimum within a square σ of side length $3\text{per}(R)$ concentric to R.

Step 2: Use a grid polygon

Use grid of size
$O(1/\varepsilon) \times O(1/\varepsilon)$
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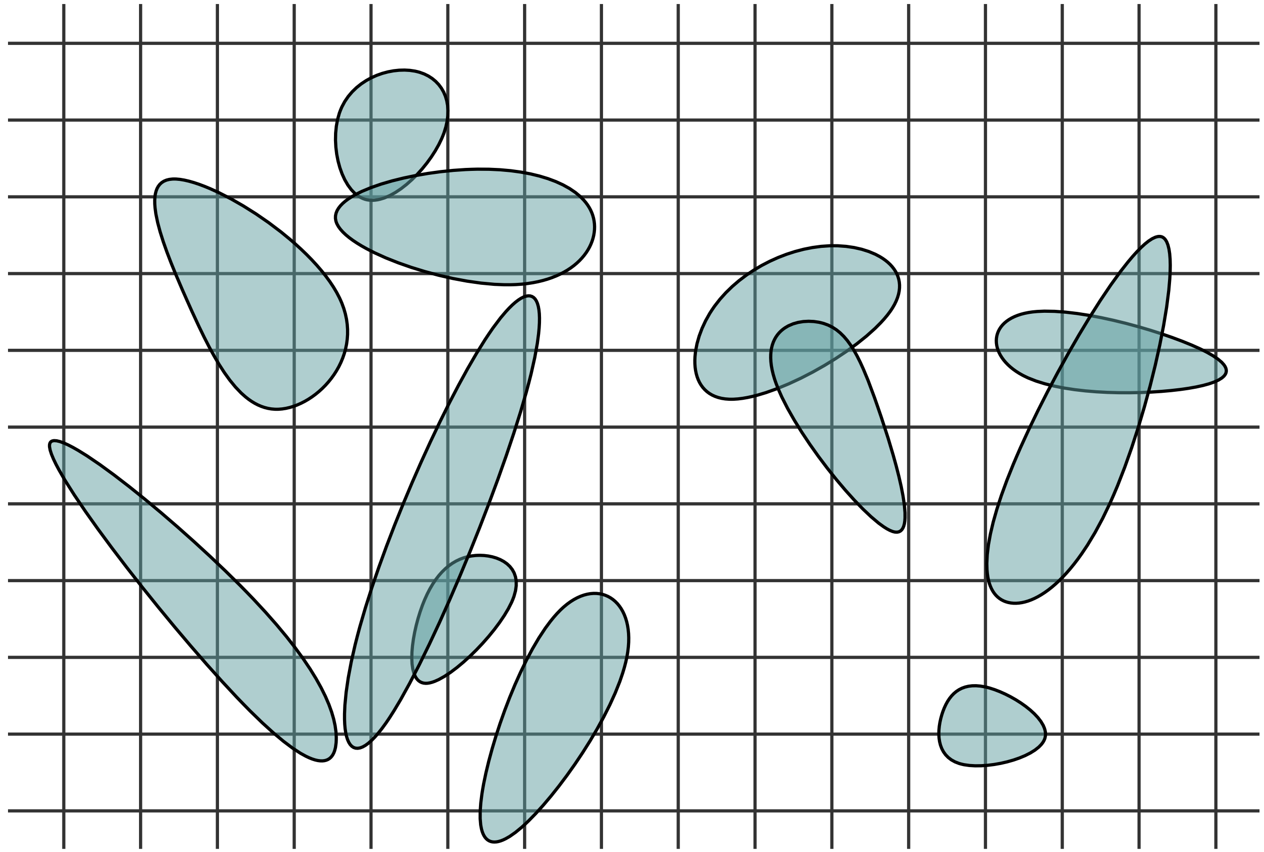

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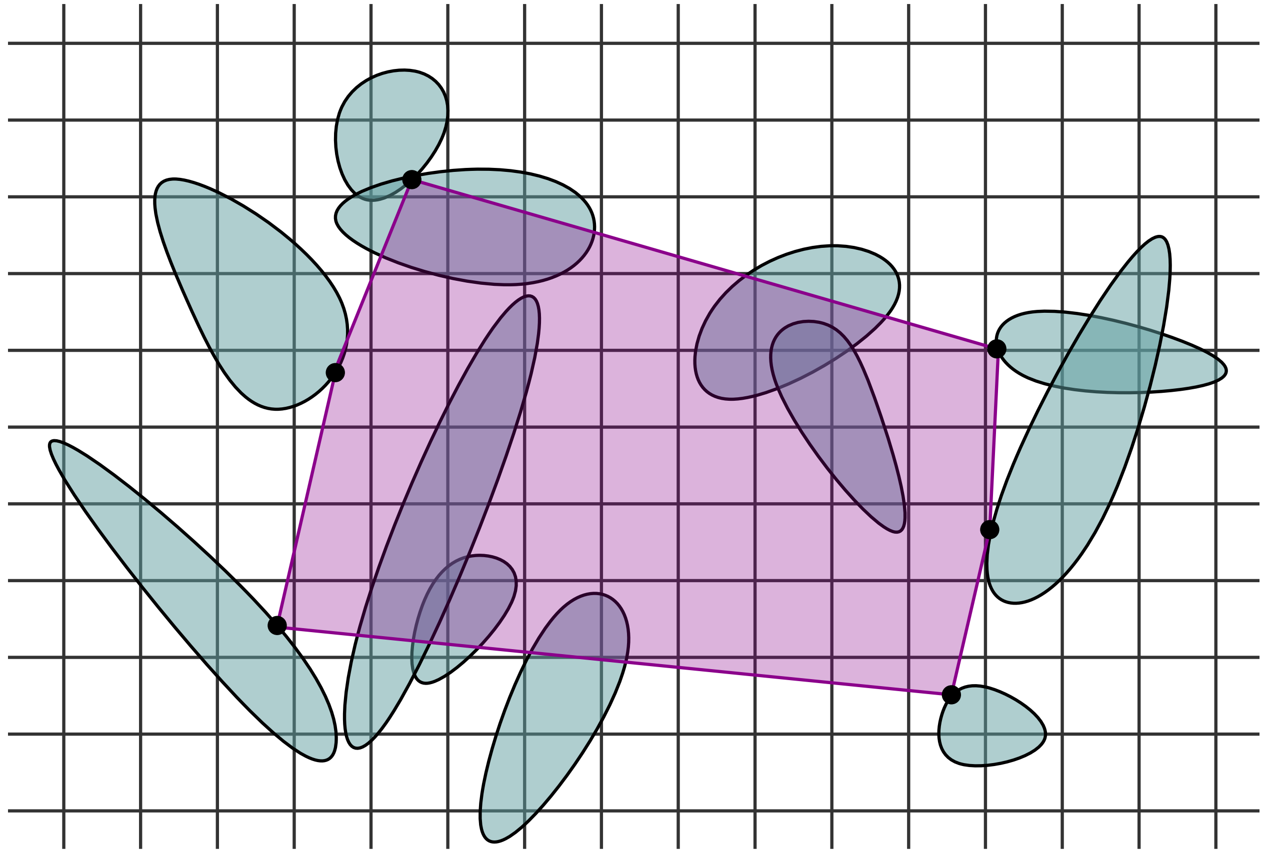

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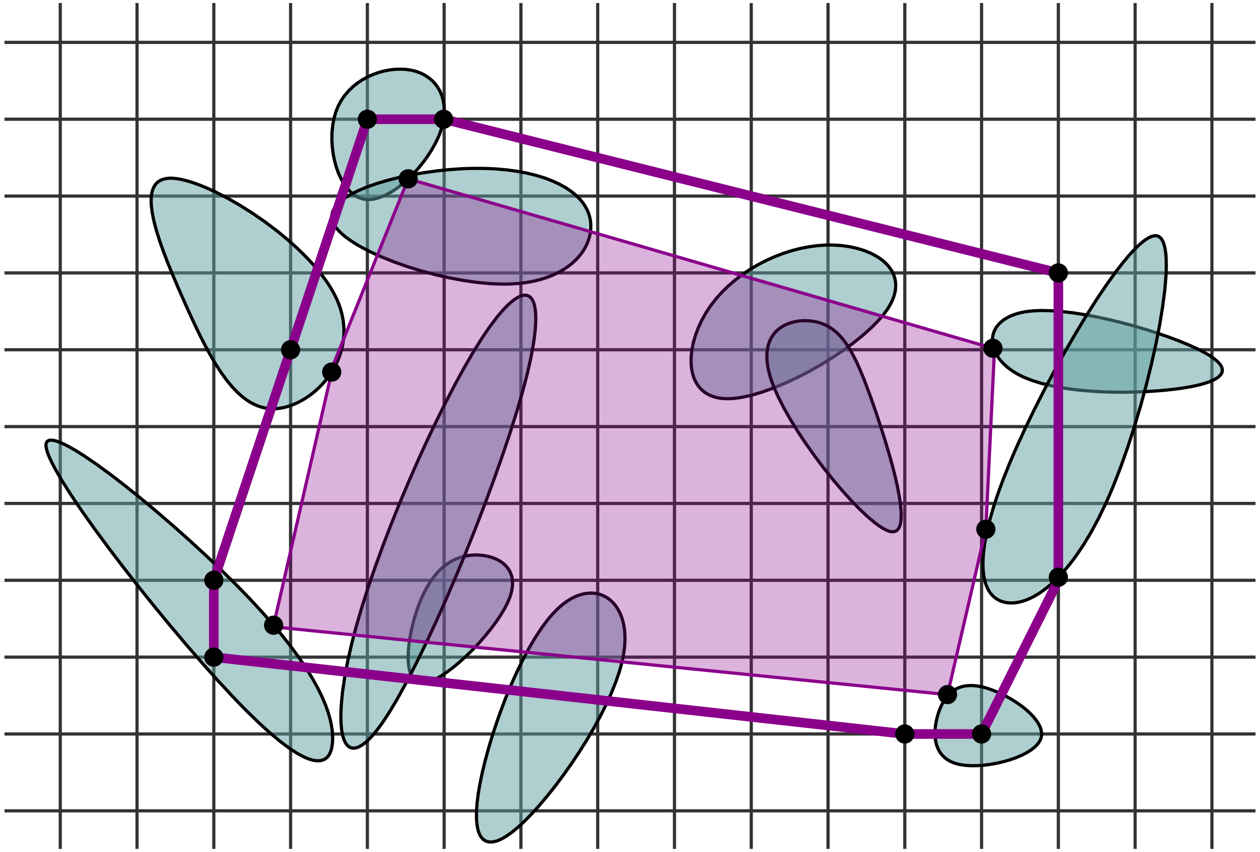

Observation

The convex hull of the grid cells containing the vertices of OPT has perimeter at most $(1 + \varepsilon)$ times bigger.

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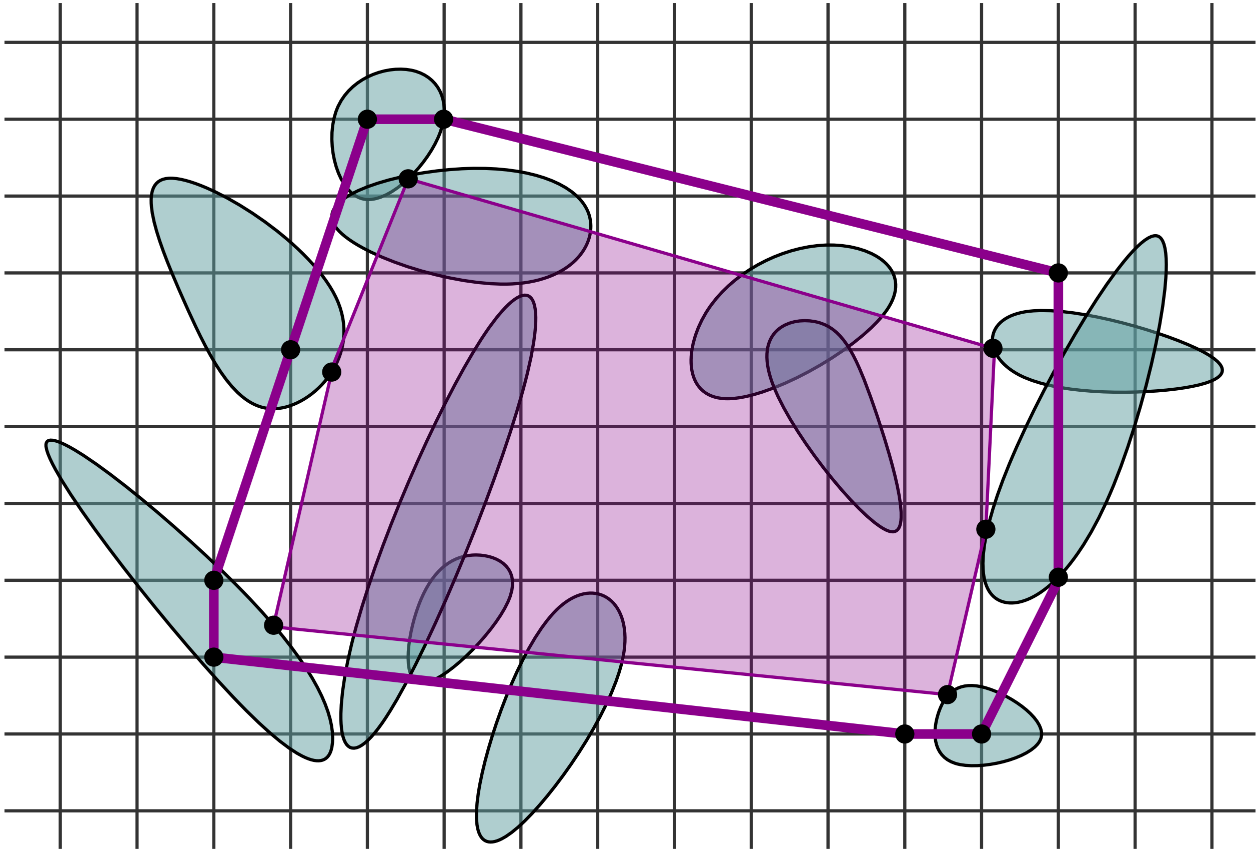

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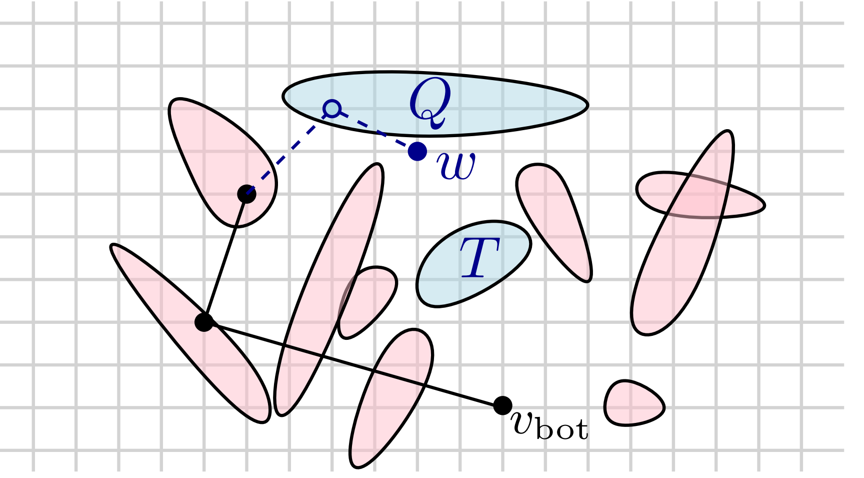
The convex hull of the grid cells containing the vertices of OPT has perimeter at most $(1+\varepsilon)$ times bigger.

It is sufficient to compute the optimum grid polygon.

Step 3: Dynamic programming, but how?

Try to compute the shortest convex chain on grid pts intersecting the objects.

But exactly which objects?



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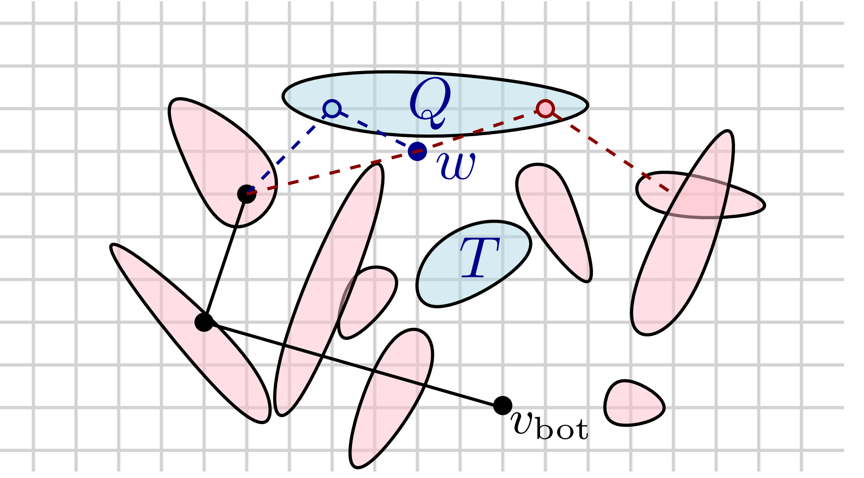
When going through w, should Q be visited before, or after w?

How do we make sure that T is included somewhere?



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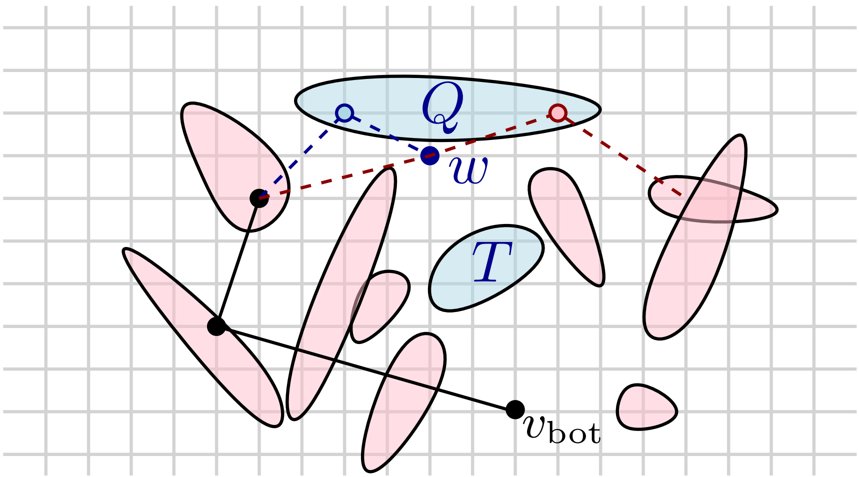

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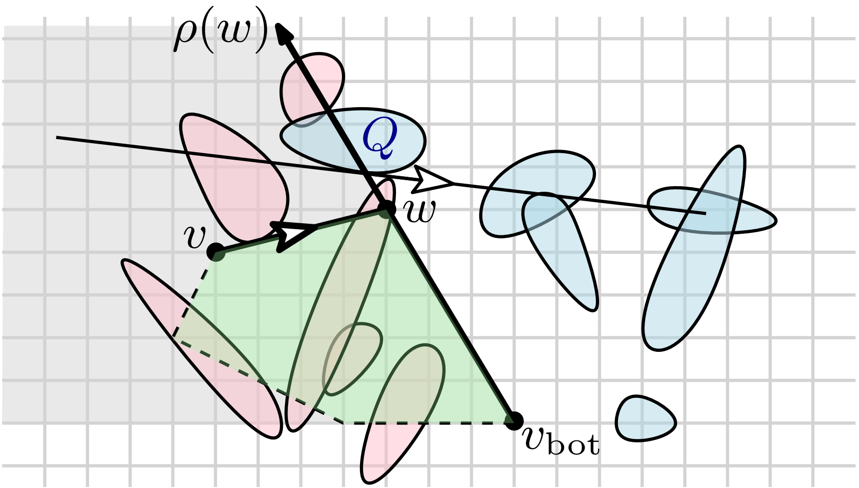
vw: last edge of current chain



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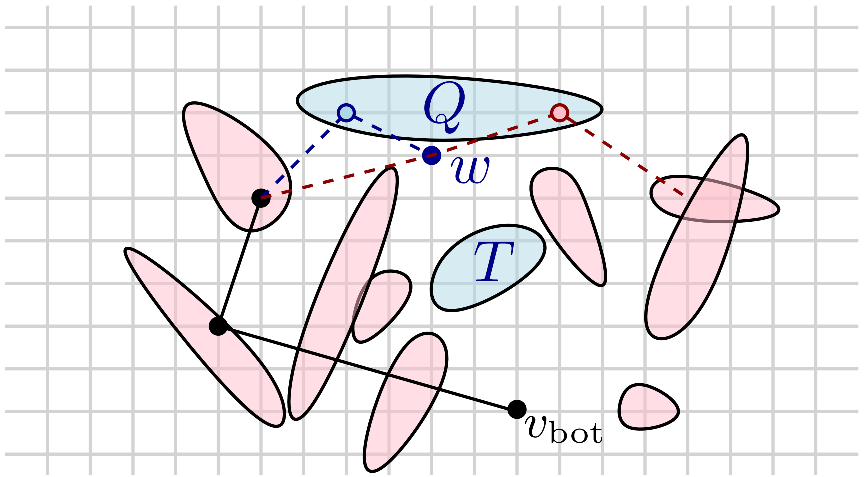

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visit X with $v_{\text{bot}}w$ if:

- X disjoint from $\rho(w)$ and is on its "left"
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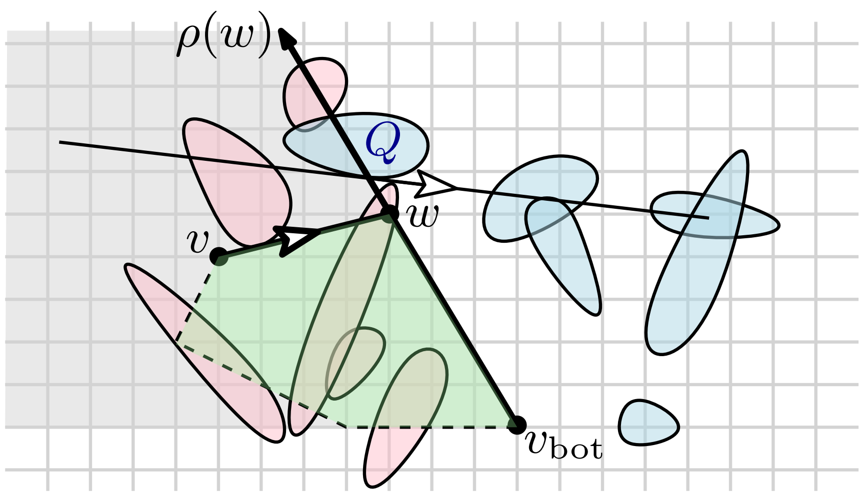

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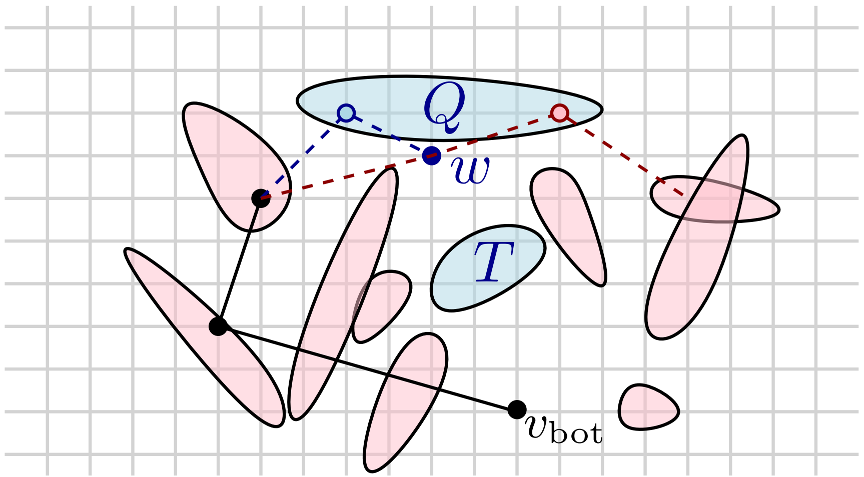

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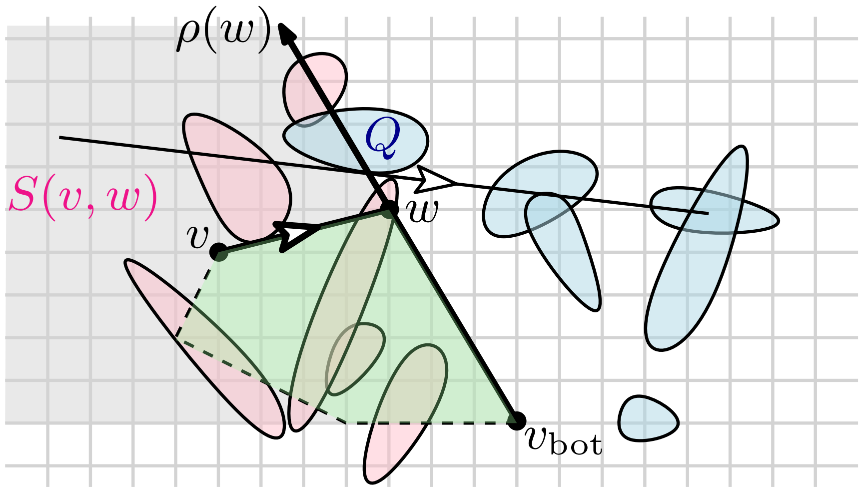

Min perimeter FPTAS wrap-up

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(from $O(1/\varepsilon) \times O(1/\varepsilon)$ grid)

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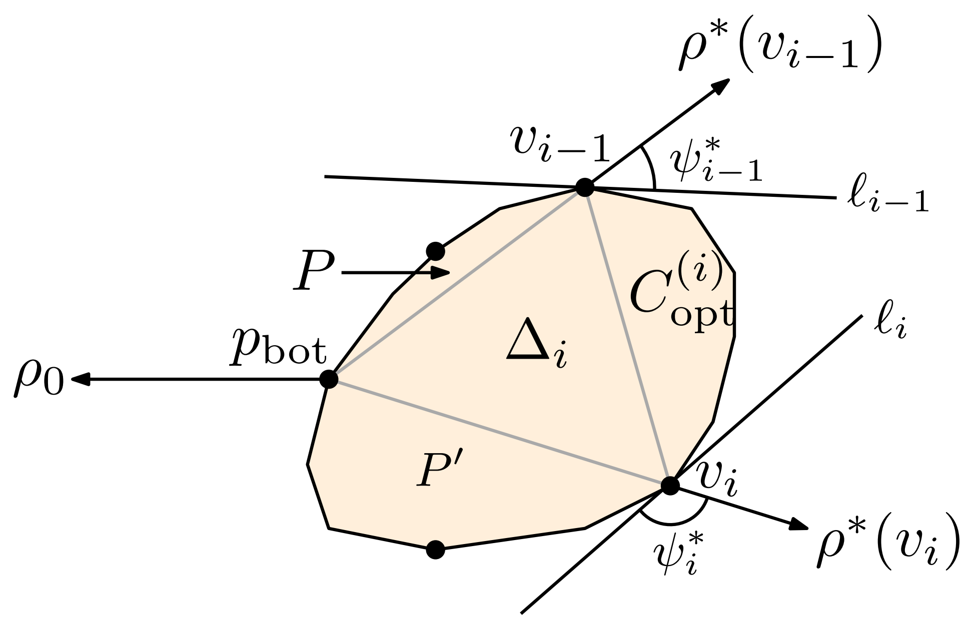
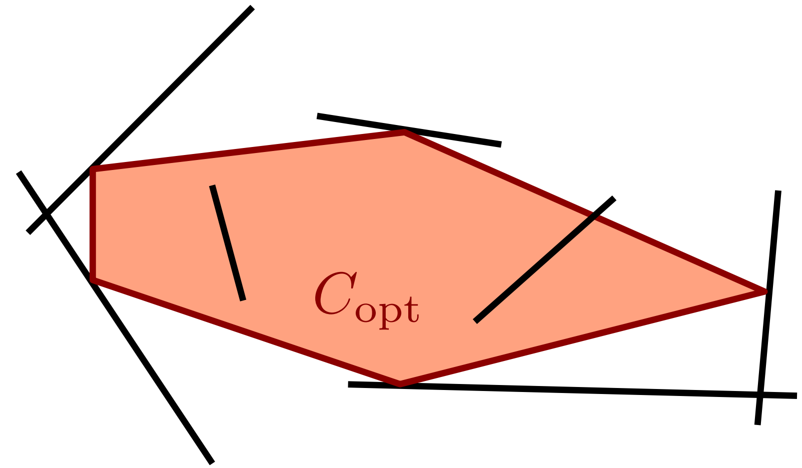
Exact algorithm for min perimeter intersecting polygon of segments



Floating tours and "orderings"

Some or all vertices of OPT are not segment endpoints!

No way to discretize!



Lemma (Dror et al.)

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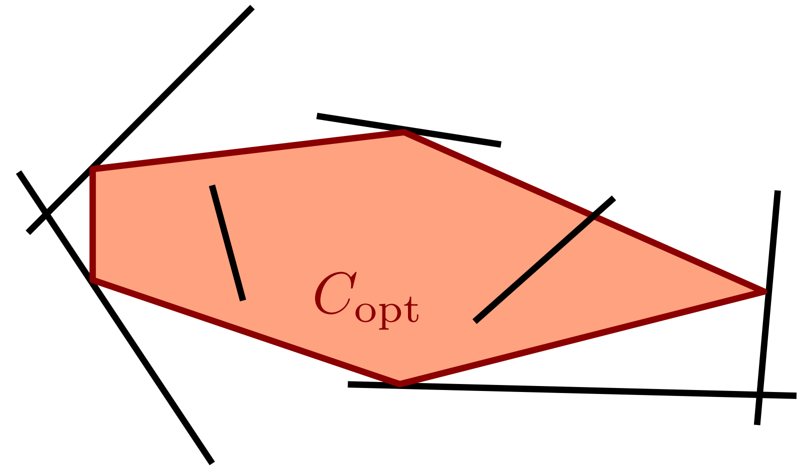

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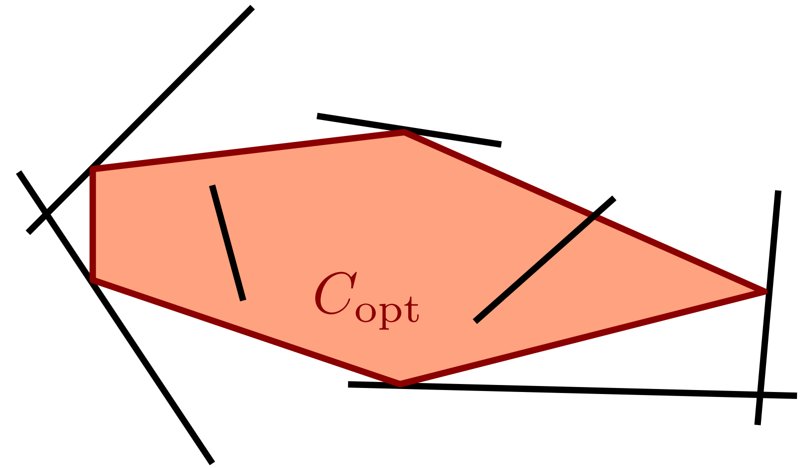

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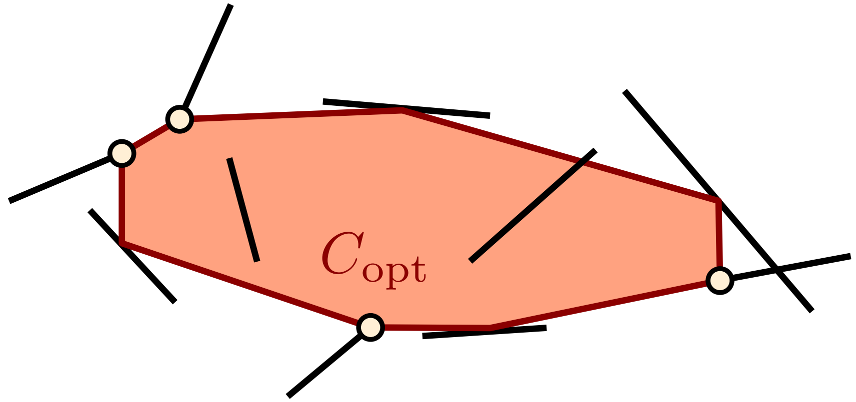
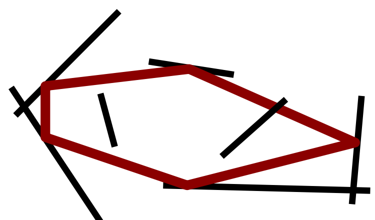
Any interval of p_i's may coincide.

Gridless DP with subroutine for floating sections

Idea: DP where we 'jump' between neighboring segment endpoints of OPT



Two subroutines:



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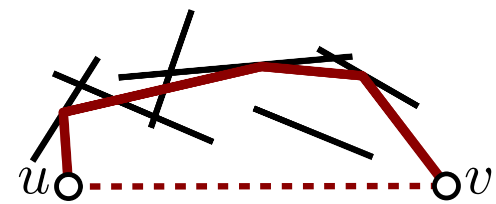

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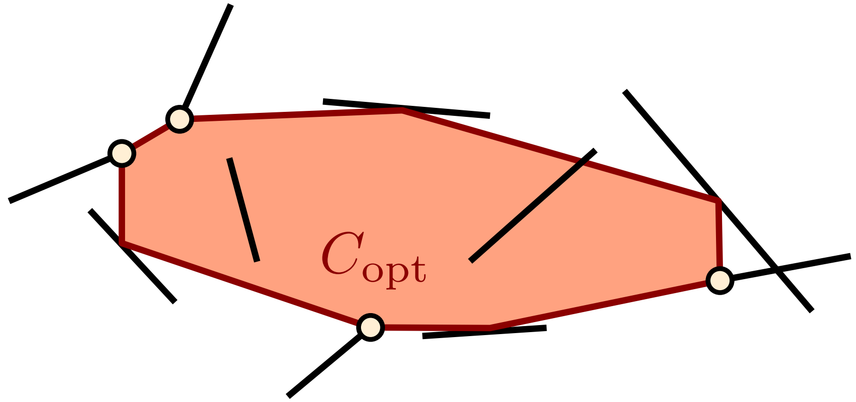

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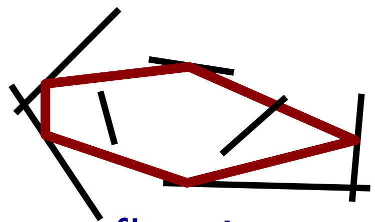

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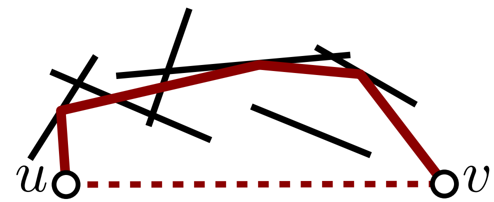

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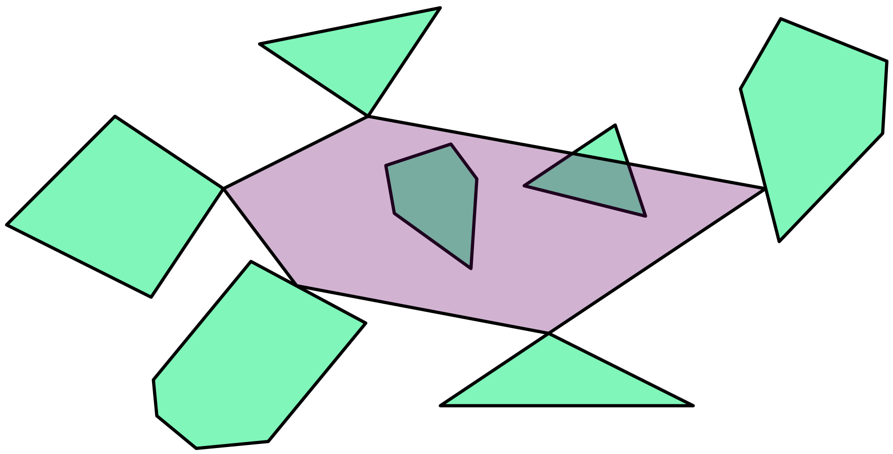
Lemma

Given u, v and k segments S, there are k half-planes bounded by segments of S named $h_1, \ldots, h_k$ in angular order of their normals, s.t. OPT visits these half-planes in order. The half-plane optimum for $u, h_1, \ldots, h_k, v$ by Dror et al. is also feasible for the segments.

Conclusion

Results:

- FPTAS for min perimeter
  int. pol. of convex objects
- exact polynomial for min perimeter
  int. pol. of line segments
- FPTAS for min area
  convex int. pol. of convex polygons



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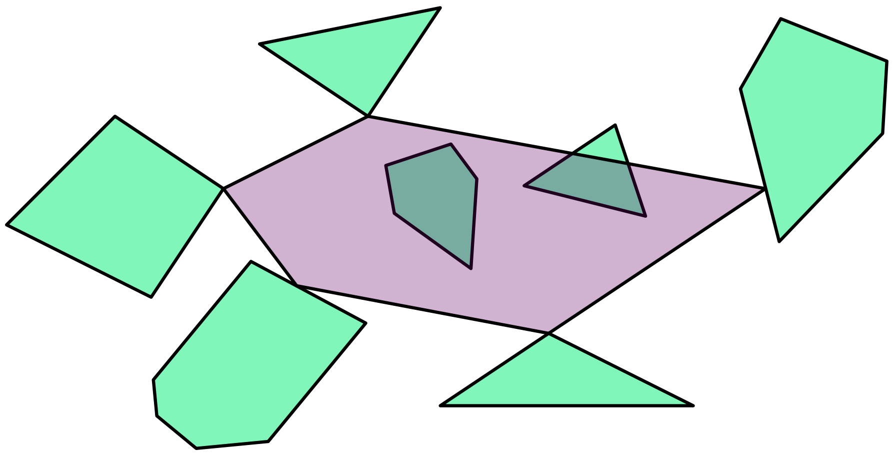

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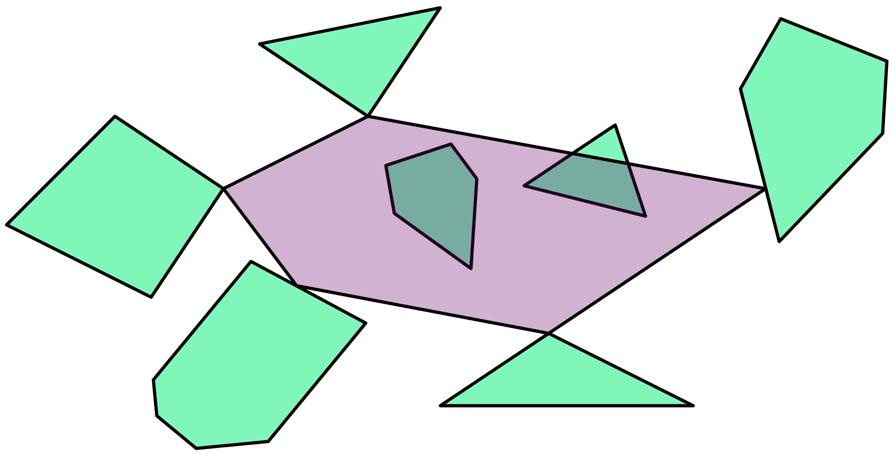

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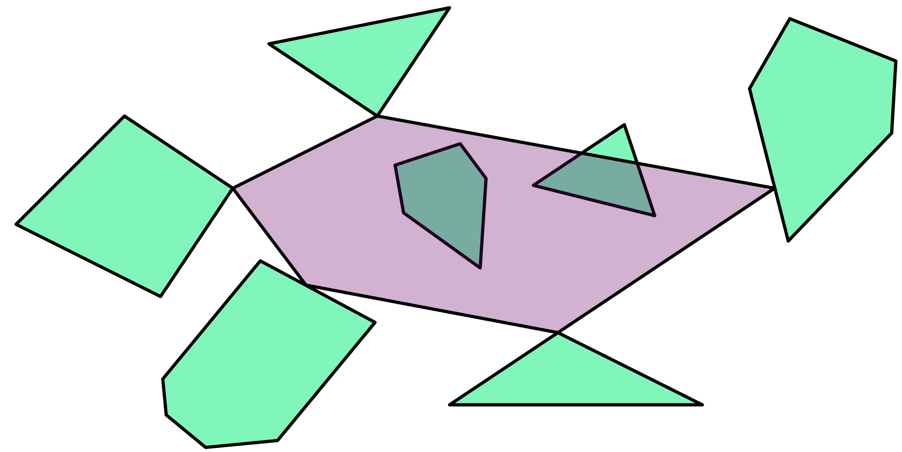

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- PTAS for minimum volume and minimum surface area polyhedron in $\mathbb{R}^d$?

Conclusion

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  int. pol. of convex objects
- exact polynomial for min perimeter
  int. pol. of line segments
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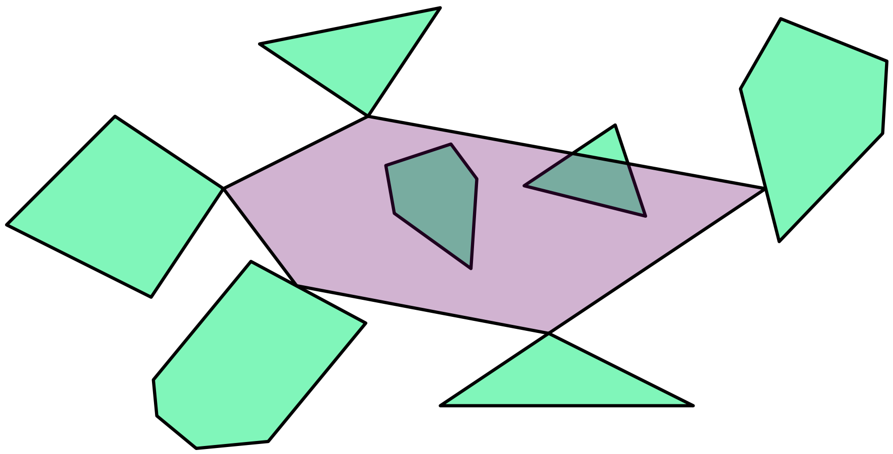

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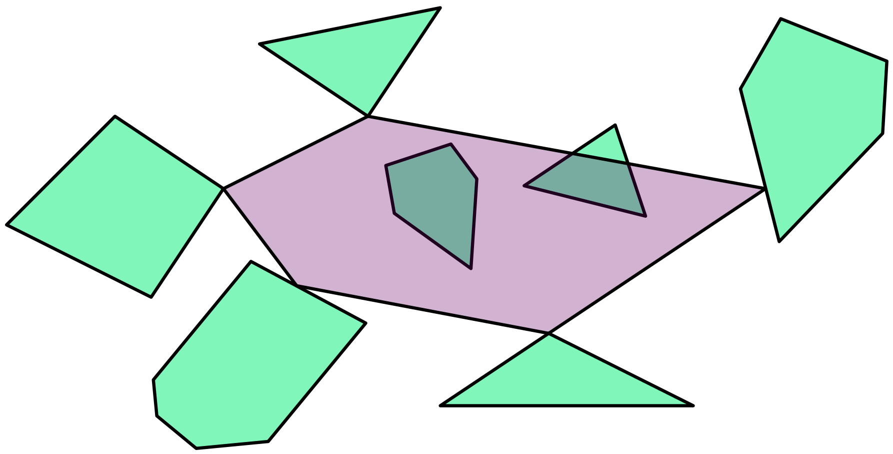

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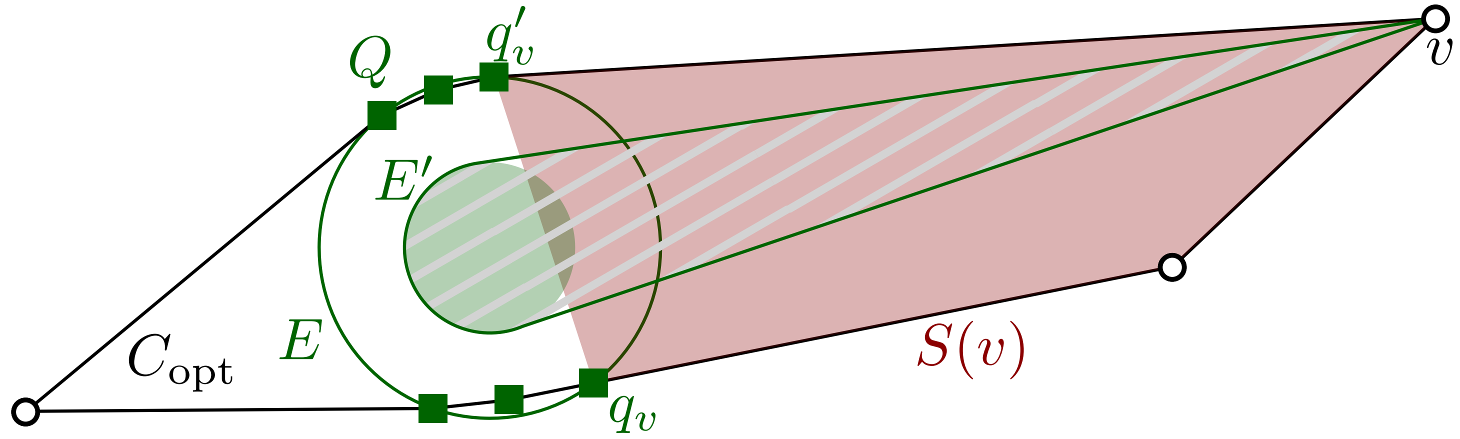
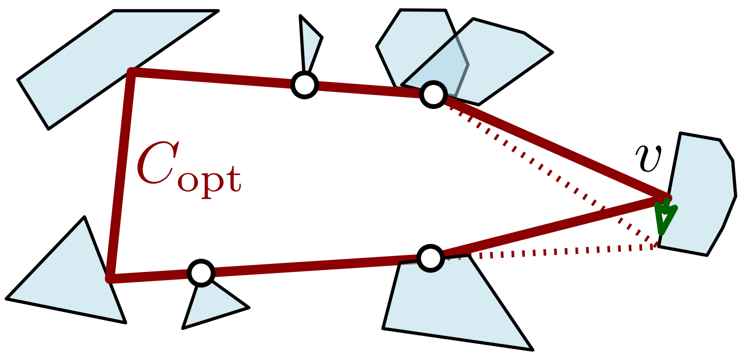
FPTAS for min area convex intersecting polygon: the spiky case





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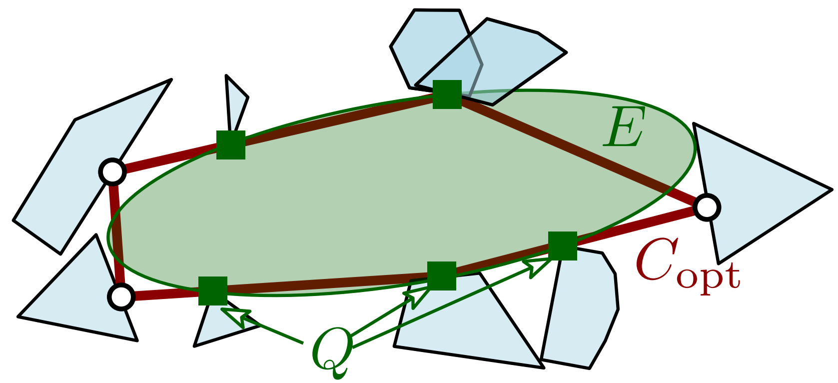

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